Lancaster Particle Pamphlets

Number 1 Relativity Formulae

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Relativity Formulae Contents

Relativity Formulae

Introduction

In this section, you will learn to use some of the basic equations of special relativity theory. These equations include new expressions for the total energy, the kinetic energy and the momentum of particles. You will also learn new expressions for conservation of energy and momentum. It is most important to realise that the equations you have already learned in Newtonian Mechanics are not “wrong”. They are approximations that are quite adequate in our everyday world. However, in particle physics where the speeds of particles often approach the speed of light, we have to use the correct equations. There is also the new concept of energy mass equivalence. This is encapsulated in the famous formula: $E = m_0 c^2$

The approach to the speed of light

The speed of light measured in vacuum is very close to $3.0\times10^8\ \mathrm{m\,s^{-1}}$. It is a basic law of relativity that the speed of a particle relative to an observer must be less than this value. We usually reserve the symbol “*c*” to denote the exact value of the speed of light in vacuum so that if *v* is the speed of a particle relative to any observer then:

$$v < c$$

[A more accurate value for *c* is $2.9979\times10^8\ \mathrm{m\,s^{-1}}$]

Here I have used the word *particle* to describe an object with **mass** such as an electron or a proton. There are also particles with zero mass! An example is the **photon**, the particle of light. All particles with zero mass travel at the speed of light and, of course, the photon is no exception.

As the speed of a particle approaches the speed of light physics becomes very interesting and it is useful to define the “gamma factor” which tells us how close we are. The gamma factor (the Greek symbol g) is defined as

$$g = \frac{1}{\sqrt{1 - v^2 / c^2}} \qquad (1)$$

[The gamma factor is a pure number with no units.]

Note that if $g = 1$ then $v = 0$. As *v* approaches *c* then *g* gets larger and larger without limit.

Exercises. 1) Use a calculator to work out g for v =0.01c, 0.1c, 0.5c, 0.9c, 0.99c. Now rearrange formula 1 to show that $v/c = \left(\sqrt{g^2 - 1}\right)/g$

2) Find v/c when g =1.1, 2.0, 10, 100.

Energy and momentum

You are familiar with the notion of *kinetic energy* in Newtonian mechanics. For a particle of mass m, this is written as $\frac{1}{2}mv^2$ and it is the energy of a particle due to its speed v. Relativity theory shows that this is only an approximation to the truth. The correct expression for the **total energy** E of a particle is

$$\boxed{E = g\, m_0 c^2 \qquad (2)}$$

We use the words *total energy* for good reason. There are two contributions to the energy of a moving particle:

1) The energy due to the mass of the particle itself; this is given by Einstein's celebrated formula $m_0 c^2$. We call this part the **rest mass energy** and m_0 is called the **rest mass**.
2) The kinetic energy or energy due to the motion of the particle.

We will see later that the two forms of energy are interchangeable. Rest mass energy can be converted into particle kinetic energy and kinetic energy into rest mass energy. This sort of thing happens all the time in particle physics! Let's now find the correct expression for the kinetic energy. The total energy is the sum of $m_0 c^2$ and the kinetic energy E_{kin}.

$$E = m_0 c^2 + E_{\text{kin}}$$

but from equation (2), $E = g\, m_0 c^2$

therefore $E_{\text{kin}} = g\, m_0 c^2 - m_0 c^2 = (g - 1) m_0 c^2$

The correct expression for kinetic energy

$$E_{\text{kin}} = (g - 1) m_0 c^2 \qquad (3)$$

Notice that the total energy of a particle with rest mass m_0 increases linearly with the gamma factor. From now on, we will refer to *total energy* simply as "the energy".

In appendix 1, we show that the Newtonian expression for kinetic energy $\frac{1}{2}m_0v^2$ is a good approximation to the correct expression (3) when the speed of the particle is very much less than the speed of light.

$$\text{When } v << c, E_{\text{kin}} = (g - 1)m_0c^2 \approx \frac{1}{2}m_0v^2$$

In Newtonian mechanics the expression for the momentum of a particle rest mass m_0, moving with speed v, is $p = m_0v$. This is also an approximation valid only when v is much less than c. The correct expression involves the gamma factor again.

The correct expression for momentum

$$p = g\, m_0v \qquad (4)$$

Look carefully at the equations (3) and (4) for energy and momentum. You will notice that a single expression can be obtained by eliminating the speed v. The new expression is an important relation between energy, momentum and the rest mass. If you are interested, the necessary algebra is given in appendix 2.

The relation between energy, momentum and rest mass

$$E^2 = p^2c^2 + m_0^2c^4 \qquad (5)$$

This formula can be rearranged to give the rest mass of a particle in terms of E and p.

$$m_0 = \sqrt{E^2 - p^2c^2}\Big/c^2 \qquad (6)$$

Particle physicists often use this formula when they have to find the mass of a new particle when they have measured its momentum and energy. This relation remains valid whatever the speed of the particle relative to the observer who is making the measurements of E and p. For this reason the rest mass m_0 is called an *invariant.*

Units for energy, momentum and rest mass

The standard SI unit for energy is the Joule. However, in atomic and nuclear physics this unit always gives awkward powers of ten. For example typical energies in atomic physics are of the order 10^{-19} J and typical nuclear energies are of order 10^{-13} J. It is more convenient to use the eV unit.

$$\boxed{\begin{gathered}1\,\text{eV} = 1.6\times10^{-19}\ \text{J}\\ 1\,\text{MeV} = 1.6\times10^{-13}\ \text{J}\\ 1\,\text{GeV} = 1.6\times10^{-10}\ \text{J.}\end{gathered}}$$

In particle physics, the energies involved are usually greater than 1 GeV and the very latest accelerators will produce beams of particles with energies greater than 1 TeV (or 1000 GeV).

We have already emphasised the importance of equation (5) giving the relationship between energy, rest mass and momentum. Notice that m_0c^2 and pc must also have units of energy and it is convenient again to use the GeV unit. In other words we express energy, momentum and mass in terms of GeV units as follows:

Quantity	**Unit**
Energy	GeV
Momentum	GeV/c
Mass	GeV/c^2

With this system, we do not have to worry about the actual value for the speed of light. In most calculations, for example using equation (5), the factors of c simply cancel out!

As an example, we can calculate the rest mass of a proton in GeV/c^2 units. The mass of a proton is 1.67×10^{-27} Kg. We have then:

$$\begin{aligned} m_0c^2 &= 1.67\times10^{-27}\times\left(3.0\times10^{8}\right)^2 = 1.50\times10^{-10}\ \text{J}\\ &\quad\text{converting to GeV we get}\\ m_0c^2 &= \left(1.5\times10^{-10}\right)/\left(1.6\times10^{-10}\right) = 0.938\,\text{GeV}\\ \text{and } m_0 &= 0.938\,\text{GeV/c}^2\end{aligned}$$

It is useful to know, and easy to remember, that the rest mass of a proton is close to 1 GeV/c^2

Conservation of energy and momentum

There are two fundamental principles to apply in the basic interactions between particles. These are the **conservation of energy** and the **conservation of momentum**. Imagine the collision of two particles during which an interaction between the particles takes place. The result of the interaction could be that only the original particles emerge, this is called an *elastic collision.* However, it is often the case that one or more additional particles are produced; this is called an *inelastic collision*. In some cases the original particles disappear altogether! In both elastic and inelastic collisions, the laws of energy conservation and momentum conservation apply.

Conservation of total energy

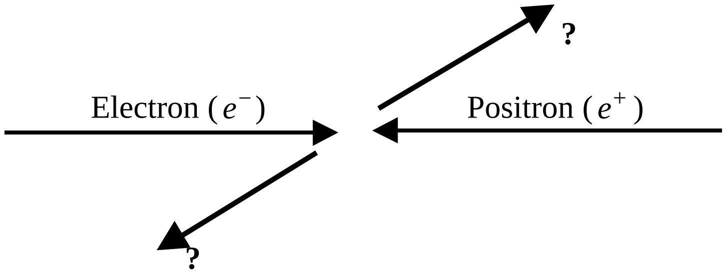
The sum of the energies of the particles present before the collision is equal to the sum of the energies of all the particles present after the collision. Remember that energy is a scalar quantity and the energies add up like ordinary numbers.

Conservation of momentum

The sum of the momenta of the particles present before the collision is equal to the sum of the momenta of all the particles present after the collision. Remember that momentum is a vector quantity and account must be taken of the direction of travel of each particle. For example, two particles of equal rest mass have zero total momentum when travelling in opposite directions with the same speed.

Using the formulae

We can illustrate the use of these relativity formulae by describing an experiment in particle physics. A **colliding beam** machine produces electrons and positrons with energy 2.50 GeV. The beams are directed towards each other with equal and opposite momenta. Occasionally an electron and positron collide and annihilate each other to produce two new charged particles. This is illustrated in Fig 1. After one such collision, the experimental team were able to find the momenta of the new particles by measuring their curvature in a magnetic field. The result was 2.32 GeV/c for each. What were the new particles?



To understand the experiment properly you have to remember that the positron is the antiparticle to the electron. It is identical in all respects except that the charge is +1 unit rather than –1 unit. (The lepton numbers are $L_e = +1$ for the electron and $L_e = -1$ for the positron). The particle and antiparticle can then annihilate and in this case produce a "blob" of energy in the form of a *virtual photon* which very quickly materialises as the two new observed particles.

Let’s first calculate the gamma factor and the momentum of the electron and positron. (This information is interesting but not actually required to work out the problem). Using equation (2) we find

$$\begin{aligned} g &= E / m_0 c^2 \\ m_0 c^2 &= 0.51\,\text{MeV and } E = 2.5\,\text{GeV for the electron, so} \\ g &= 2.50\times10^9 / 0.51\times10^6 = 4.9\times10^3 \end{aligned}$$

We see that the gamma factor is very large because the electron rest mass energy is much smaller than its total energy. You can now check that the numerical values for the electron momentum (in GeV/c) and for the kinetic energy (in GeV) are very close to 2.5.

Next we calculate the sum of energies of the electron and positron before the collision. This comes to 2.5 + 2.5 = 5.0 GeV. The net momentum before the collision is zero, because the electron and positron are moving towards each other with equal magnitude of momentum. Conservation of energy says that the energy of the new particles after the collision must be equal to the energy before, i.e. 5.0 GeV. Conservation of momentum says that the total momentum after the collision is zero.

The two produced particles appear to have the same momentum and they must move off in opposite directions to conserve momentum. If the produced particles are a particle and antiparticle pair then their masses will be equal. It follows from equation (5) that their total energies must also be equal. This in turn implies that the new particles each has $E = 5.0/2 = 2.5\,\text{GeV}$.(In practice the experimenters would check that the produced particles did move off in exactly opposite directions, if they did not then an additional unrecorded particle must have been produced.)

We now use equation (6) to evaluate the rest mass of the new particles since we know the total energy and momentum of each one.

$$\begin{aligned} m_0 &= \sqrt{E^2 - p^2c^2} \big/ c^2 \\ E &= 2.5\,\text{GeV and } p = 2.32\,\text{GeV/c} \\ m_0 &= \sqrt{2.50\times2.50 - 2.32\times2.32} \big/ c^2 \\ m_0 &= 0.938\,\text{GeV/c}^2 \end{aligned}$$

[Notice again that in our new units for energy, momentum and mass we do not need to know the value of *c*.]

The mass of each created particle is 0.938 GeV/c^2. This identifies them as a **proton** and **antiproton** pair. Careful observation by the experimental team of the direction of curvature of the particles in the magnetic field would confirm that the charges had opposite sign.

The gamma factor of the created proton (or antiproton) may now be calculated from equation (2).

$$\begin{aligned} g &= E / m_0 c^2 \\ g &= 2.5\,(\text{GeV}) / 0.938\,(\text{GeV}) = 2.67 \end{aligned}$$

Compare this with the gamma factor of the 2.5 GeV electrons in the beam from the accelerator. The produced proton and antiproton have the same energy as the beam particles but the gamma factor is much smaller since their masses are much greater. The kinetic energy of the proton is given by equation (3):

$$\begin{aligned} E_{\text{kin}} &= (g - 1) m_0 c^2 \\ E_{\text{kin}} &= (2.67 - 1) \times 0.938 \\ &= 1.57\ \text{GeV} \end{aligned}$$

The gamma factor also gives the speed as a fraction of the speed of light. Starting with equation (1)

$$g^2 = 1 \big/ (1 - v^2 / c^2)$$

rearrange this equation to get

$$\left(\frac{v}{c}\right)^2 = 1 - \frac{1}{g^2}$$

$$\text{Hence}\ \frac{v}{c} = \sqrt{g^2 - 1} \big/ g$$

Substitute $g = 2.67$:

$$\frac{v}{c} = 0.927$$

This example illustrates one of the advantages of a colliding beam machine. A large fraction of the initial energy of the colliding beams goes into creating the rest mass of the new particles. In fact we could reduce the colliding beam energies to 0.938 GeV and still just be able to produce the proton anti proton pair. [There will be further discussion on the advantages and disadvantages of colliding beam and fixed target machines in the pamphlet on particle accelerators.]

Appendix 1

Often in physics, one comes across expressions of the form $y = (1+x)^n$ with $|n| < 1$. Using the binomial expansion $y = 1 + nx + \frac{n(n-1)}{2!} + \ldots$ we see that, when $x << 1$, $y \approx 1 + nx$ is a good approximation. Starting with equation (3) for the kinetic energy:

$$\begin{aligned} E_{\text{kin}} &= (g - 1) m_0 c^2 \\ &= \left(\left(1 - v^2/c^2\right)^{-1/2} - 1 \right) m_0 c^2 \end{aligned}$$

$$\text{but } \left(1 - v^2/c^2\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \text{ when } \frac{v^2}{c^2} << 1$$

$$\text{Hence } E_{\text{kin}} \approx \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1 \right) m_0 c^2 = \frac{1}{2} m_0 c^2$$

When the speed of a particle is very much less than the speed of light, the correct expression (3) for the kinetic energy reduces to the familiar Newtonian expression.

Appendix 2

Equation (5) is derived from equations (2) and (4):

$$E = g\, m_0 c^2 \qquad (2)$$

$$p = g\, m_0 v \qquad (4)$$

$$\begin{aligned} E^2 + p^2 c^2 &= g^2 \left(m_0 c^2\right)^2 + g^2 m_0^2 v^2 c^2 \\ &= g^2 \left(m_0 c^2\right)^2 \left(1 + \frac{v^2}{c^2} \right) \end{aligned}$$

$$\text{but } g^2 = \left(1 + \frac{v^2}{c^2} \right)^{-1}$$

$$\text{therefore } E^2 + p^2 c^2 = \left(m_0 c^2\right)^2 = m_0^2 c^4$$

This is the desired result. An expression relating the energy E, the momentum p and the rest mass m_0.