Nonexponential Decoherence and Momentum Subdiffusion in a Quantum Lévy Kicked Rotator

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We investigate decoherence in the quantum kicked rotator (modeling cold atoms in a pulsed optical field) subjected to noise with power-law tail waiting-time distributions of variable exponent (Lévy noise). We demonstrate the existence of a regime of nonexponential decoherence where the notion of a decoherence rate is ill defined. In this regime, dynamical localization is never fully destroyed, indicating that the dynamics of the quantum system never reaches the classical limit. We show that this leads to quantum subdiffusion of the momentum, which should be observable in an experiment.

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The transition between quantum and classical dynamics is not yet fully understood and the boundary between the two remains fuzzy. Deep insight into the emergence of classical properties from quantum mechanics is provided by the theory of decoherence [1,2]. In this framework, the unavoidable coupling to an external environment induces a dynamical suppression of interference effects between states of a quantum system, and thus classical behavior. The disappearance of quantum interference can often be described by a decoherence factor of the form $\mathcal{D}(t) = \exp(-t/t_c)$. For a diffusing point particle, the coherence time $t_c = 1/[m\kappa(\Delta x)^2]$ depends on the mass m of the particle, the coupling strength κ, and the spatial separation Δx of the interfering states [1,2]. The functional form of the decoherence factor, in particular, its exponential time dependence, has been verified in many experiments [3–6]. The boundary between the quantum and the classical world can accordingly be explored by properly tuning the value of the coherence time t_c. This has been performed in various ways, either by changing the separation Δx as in the QED cavity and ion trap experiments by Brune *et al.* [3] and Myatt *et al.* [4], or by modifying the value of the coupling constant κ as done in the atom-optics experiments by Ammann *et al.* [5] and Klappauf *et al.* [6]. In yet another approach, the quantum-classical border has been investigated by increasing the mass m of the system as in the double-slit experiments by Hackermüller *et al.* [7].

Common to all these studies is the exponential functional dependence of the decoherence factor. Our aim in this Letter is to widen our understanding of the quantum-to-classical crossover by proposing an environmental coupling scheme for the atom-optical experiments [5,6] which allows the controlled tuning of the functional form of $\mathcal{D}(t)$ itself, in particular, of its time dependence, and not just of its parameters. In this way, we are able to uncover some hidden aspects of decoherence, especially of its interplay with complex quantum dynamics [8].

A paradigm of complex quantum dynamics is cold atoms exposed to pulsed optical fields [9]. For chaotic conditions the classical motion in these systems exhibits momentum diffusion which, however, is suppressed in the quantum regime because of dynamical localization [10]. In the recent decoherence experiments [5,6], the quantum-classical transition has been investigated by varying the amplitude of noisy pulses acting on the atoms (coupling strength κ in t_c). This then results in a nonvanishing quantum momentum diffusion with a renormalized diffusion constant. By contrast, we here propose to modify the length τ of the time intervals between the noisy pulses [11]. Specifically, we shall consider that the intervals are generated in a renewal process with a waiting-time distribution $w(\tau)$ that asymptotically behaves as a power law, $w(\tau) \propto \tau^{-1-\alpha}$. Power-law tail waiting-time distributions of this kind—generally referred to as Lévy statistics [12]—occur naturally in many physical systems: Two prominent examples being subrecoil laser cooling [13] and fluorescent intermittency of nanocrystal quantum dots [14] (in both these systems, α is close to one-half). In the atom-optical setting, the statistics are under the control of the experimentalist. By tuning the value of the exponent α, an additional parameter not present in former decoherence studies, one is able to control the duration between successive random kicks given by the mean waiting time, $\bar{\tau} = \int_0^\infty d\tau\tau w(\tau)$. This offers a unique opportunity to smoothly interpolate between a fully coupled situation with many kicks (large α) to an almost isolated situation with very few kicks (small α). Lévy kicked systems have lately been investigated at the classical [15] as well as the quantum-mechanical level [16] and unusual properties such as nonexponential behavior and aging have been found.

Interestingly, for $\alpha \leq 1$, the mean waiting time $\bar{\tau}$ becomes infinite. We show that this divergence dramatically affects the loss of phase coherence of the atoms: the decoherence factor stops being exponential in time (instead it is given by a Mittag-Leffler function) and the concept of a decoherence rate becomes ill defined. This results in quantum subdiffusion of the momentum of the Lévy kicked rotator, a unique signature which would be directly accessible in experiments with similar parameters (e.g., pulse duration and potential depth) as for dynamical localization [5,6].

Model.—The motion of an atom in a pulsed optical field can be modeled as a kicked rotator, a connection that has been established extensively in both theoretical and experimental works [9,17,18]. The kicked rotator can be simply seen as a particle moving on a ring and periodically kicked in time. In conveniently scaled coordinates, the Hamiltonian is given by $H = p^2/2 + \sum_{n=-\infty}^{\infty} K_n \cos\theta\,\delta(t - n + 0^+)$, where the kicking potential depends on the 2π-periodic rotation angle θ. We assume that the particle is initially prepared in the zero-momentum state $\psi(0) = |p = 0\rangle$. The subsequent stroboscopic dynamics $\psi(t+1) = F(K_t)\psi(t)$ are generated by the Floquet operator $F(K_t) = \exp(-i\hbar^{-1}\hat{p}^2/2) \times \exp(-i\hbar^{-1}K_t \cos\hat{\theta})$. This operator describes a kick in which momentum changes by $K_t \sin\theta$, followed by a free rotation in which the rotation angle θ increases by p.

For constant (noiseless) kicks of strength $K_n = K \gtrsim 5$, the classical dynamics is chaotic, and on average diffusive in momentum direction: $\mathrm{var}\, p(t) \simeq D_{\mathrm{cl}} t$, where $D_{\mathrm{cl}} \simeq K^2/2$. In the quantum regime, the kicked rotator exhibits dynamical localization [10], an interference phenomenon which manifests itself in an exponentially decaying envelope of the quasienergy eigenstates of $F(k)$ in the momentum representation. As a result of dynamical localization, momentum diffusion is suppressed and the variance saturates after the quantum break time t^*,

$$\mathrm{var}\; p_0(t) \simeq D^* t^*[1 - \exp(-t/t^*)] \quad (\text{with } D^* \simeq D_{\mathrm{cl}}). \tag{1}$$

By assuming that diffusion is classical until localization sets in, the break time can be written as $t^* \simeq D^*/\hbar^2$ [18].

The dramatic difference between the classical and quantum momentum dynamics makes the kicked rotator ideally suited for investigating the crossover between both regimes. The crossover can be induced by subjecting the particle to additional random kicks—simulating in such a way the noisy coupling to an external environment [5,6]. In a noisy kick, the kicking strength is slightly perturbed away from the mean value K, $K_n = K + k_n$, where the perturbations k_n are random numbers with average $\bar{k}_n = 0$ and variance $\bar{k}_n^2 = \kappa$. The parameter κ defines the strength of the noise. The time intervals between subsequent kicks are generated by a renewal process with waiting distribution $\omega(\tau)$. Unlike the familiar Gaussian white noise, Lévy noise with a power-law waiting-time distribution is a nonstationary process with an autocorrelation function that depends explicitly on the two time variables.

The effect of noise on the quantum kicked rotator is best understood using the formalism developed by Ott, Antonsen, and Hanson [19]. In the absence of noise the quantum system is fully coherent: the quasienergy eigenstates of the Floquet operator $F(K)$ are exponentially localized in momentum space, chaotic diffusion is strongly suppressed, and the momentum diffusion constant asymptotically vanishes. The effect of the external noise is to couple the quasienergy states and to induce transitions between them. As a result, dynamical localization is extenuated and quantum diffusion takes place. There is thus an intimate connection between quantum momentum diffusion and phase coherence in the kicked rotator.

A systematic, perturbative approach for calculating the quantum diffusion constant for weak stationary noise with arbitrary correlations has been put forward by Cohen [20]. In this case, the survival probability of the quasienergy eigenstates is found to decay exponentially over time, $\mathcal{D}(t) = \exp(-t/t_c)$. In the limit of weak noise, $\kappa \ll K^4/\hbar^4$, the coherence time is given by $t_c \simeq 2\hbar^2/\kappa$. The resulting momentum spreading is accordingly [20]

$$\mathrm{var}\; p \simeq \frac{D^*}{1 + t_c/t^*} t + \frac{D^* t^*}{(1 + t^*/t_c)^2}[1 - \exp(-t/t^* - t/t_c)]. \tag{2}$$

Hence, for stationary noise when quantum coherence is lost exponentially, the momentum diffuses asymptotically with a renormalized diffusion constant $D^*/(1 + t_c/t^*)$. Equation (2) has been successfully applied to quantify the decoherence process in the experiment [6].

Momentum spreading.—In the following, we extend the calculation of the quantum diffusion constant beyond the perturbative regime: for the present case of uncorrelated perturbations κ_n, a random-phase approximation allows us to study decoherence induced by *nonstationary* noise (for details see the end of this Letter). In a general renewal process, the number of noisy events $N(t', t'')$ within an interval $[t', t'']$ (technically known as the *inverse time* of the process) is a statistical quantity. Within the random-phase approximation, the survival probability in a quasienergy eigenstate over this time interval is simply given by the decoherence factor

$$\mathcal{D}(t', t'') = \overline{\exp[-N(t', t'')/t_c]}, \tag{3}$$

where t_c is the coherence time of the corresponding stationary noise process with the same noise intensity. The variance of the momentum is then given by

$$\begin{aligned}\mathrm{var}\; p(t) = {} & \mathrm{var}\; p_0(t)\mathcal{D}(t, 0) + \frac{\kappa}{2}\overline{N(t, 0)} \\ & + \int_0^t ds[\mathrm{var}\; p_0(t - s)\partial_s \mathcal{D}(t, s) \\ & - \mathrm{var}\; p_0(s)\partial_s \mathcal{D}(s, 0)] \\ & - \int_0^t ds' \int_0^{s'} ds''\, \mathrm{var}\; p_0(s' - s'')\partial_{s'}\partial_{s''}\mathcal{D}(s', s'').\end{aligned} \tag{4}$$

Here $\mathrm{var}\; p_0(t)$ is the variance of the momentum in the absence of noise. The second term on the right-hand side only becomes important for large noise strength.

Stationary noise can be regarded as a special case of a renewal process with $w(\tau) = \delta_{1,\tau}$. In the latter situation, the number of noisy events is simply $N(t', t'') = |t' - t''|$ and the decoherence function decays exponentially, $\mathcal{D}(t', t'') = \exp(-|t' - t''|/t_c)$. Equation (4) then provides a generalization of the perturbative result (2) which accounts exactly for all correlations of the noiseless dynamics. Expression (2) is recovered when the noiseless dynamics is approximated by the variance (1).

We now evaluate the decoherence factor (3) for a renewal process with a general waiting-time distribution. The Laplace transform of $\mathcal{D}(t, 0)$ with respect to t reads $\mathcal{D}(u, 0) = u^{-1}(1 + f(u)/t_c)^{-1}$, where $f(u) = w(u)/(1 - w(u))$ is the Laplace transform of the *sprinkling distribution* $f(t) = \partial_t \overline{N(t, t')}$ [$f(t)$ is simply the probability that there is a noise event at time t]. For two time arguments, we have (see also Ref. [16] for similar results),

$$\mathcal{D}(t', t'') = \mathcal{D}(t', 0) + t_c^{-1} \int_0^{t''} ds f(s) \mathcal{D}(t' - s, 0). \quad (5)$$

For Lévy noise with asymptotic power-law dependence $w(\tau) \sim c\tau^{-1-\alpha}$ (where c is a constant), two cases have to be distinguished: for an exponent $\alpha > 1$, the sprinkling distribution approaches a constant for long times, $f(t) \simeq \bar{\tau}^{-1}$, and the decoherence function $\mathcal{D}(t', t'') \simeq \exp[-|t' - t''|/(\bar{\tau} t_c)]$ still has an asymptotic exponential form. We thus have the important result that the effective coherence time $\bar{\tau} t_c$ is directly proportional to the mean waiting time of the noisy kicks and therefore increases with decreasing exponent α. In particular, the effective coherence time becomes *infinitely* large when α drops below unity, indicating that the classical limit cannot be attained in finite times. Clearly, the functional form of the decoherence factor has to change as well. For $\alpha < 1$, the asymptotic behavior of the sprinkling distribution is $f(t) \simeq [\alpha \sin(\pi\alpha)/(\pi c)] t^{\alpha - 1}$. The relation of its Laplace transform to $D(u, 0)$, given above, then yields the decoherence function

$$\mathcal{D}(t, 0) \simeq E_\alpha(t^\alpha / [\Gamma(-\alpha) c t_c]), \quad (6)$$

where $E_\alpha(z) = \sum_{n=0}^{\infty} z^n / \Gamma(\alpha n + 1)$ is the Mittag-Leffler function [21]. Using the known asymptotic expansions of $E_\alpha(z)$, we find that the initial decay of the decoherence function is a stretched exponential, $\mathcal{D}(t, 0) \simeq \exp\{t^\alpha / [\Gamma(-\alpha) c t_c]\}$, while for large times it crosses over to the power law $\mathcal{D}(t, 0) \simeq (c t_c / \alpha) t^{-\alpha}$. The functional dependence of the decoherence factor can thus be changed in a controlled manner by tuning the value of the exponent α. The consequences of such a modification can be directly detected in the momentum diffusion of the kicked rotator: As α approaches unity from above, the decoherence time increases, and the asymptotic quantum diffusion coefficient $D^*/(1 + \bar{\tau} t_c / t^*)$ vanishes at $\alpha = 1$. For $\alpha < 1$ it follows from Eq. (4) that the momentum spreads subdiffusively,

$$\text{var } p(t) \simeq \frac{D^* t^*}{t_c} \frac{\sin \pi\alpha}{\pi c} t^\alpha. \quad (7)$$

This is in stark contrast to the classical momentum spreading, which for noise with $\alpha \leq 1$ asymptotically reduces to the chaos-induced diffusion, $\text{var } p(t) \simeq D_{\text{cl}} t$.

The full time dependence of the momentum spreading in kicked rotators with specific implementations of Lévy noise is shown in Fig. 1. The regular kicking strength is set to $K = 7.5$, while $\hbar = 2\pi \times 577/13872$. The random numbers k_n are taken from a uniform box distribution over an interval $(-W, W)$, so that $\kappa = W^2/3$. The waiting-time distribution is of the Yule-Simon form $\omega(\tau) = \alpha \Gamma(\tau) \Gamma(\alpha + 1) / \Gamma(\tau + \alpha + 1)$, with asymptotic behavior $\omega(\tau) \sim \alpha \Gamma(\alpha + 1) / \tau^{\alpha + 1}$ [hence $c = \alpha \Gamma(\alpha + 1)$] and a mean waiting time $\bar{\tau} = \alpha / (\alpha - 1)$ [22]. In Fig. 1, the solid lines are computed by extensive numerical simulations of the dynamics of kicked rotators over 10^5 time steps, averaged over 100–1000 realizations of the noise. The dashed lines are evaluated from Eq. (4), where $\text{var } p_0(t)$ is approximated by Eq. (1) and $D^* = \hbar^2 t^* = 45.28$ is obtained by a single-parameter fit to the results of the numerical simulations without noise. The decoherence function (3) for the Yule-Simon distribution is determined setting

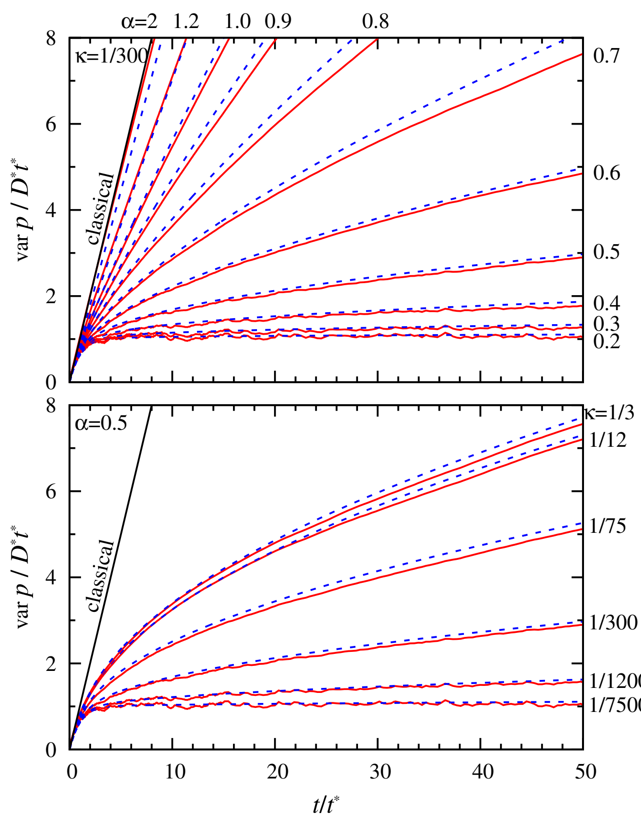

FIG. 1 (color online). Time dependence of the momentum spreading $\text{var } p(t)$ for kicked rotators which are subjected to Lévy noise generated by a Yule-Simon distribution. The results of numerical simulations (solid curves) are compared to the theoretical predictions from Eq. (4) (dashed curves). The straight solid line shows the classical diffusive momentum spreading. In the upper panel, the exponent α varies while the noise strength $\kappa = 1/300$ is fixed. The bottom panel shows results for fixed $\alpha = 0.5$ and varying noise strength κ.

$t_c = 2\hbar^2/\kappa$. We find good agreement between theory and numerics, both as a function of the exponent α as well as a function of the noise strength κ. In particular, the numerics confirm the subdiffusive long-time dynamics (7) for $\alpha \leq 1$, as well as the diffusive behavior for $\alpha > 1$.

Details of the calculation.—The derivation of our main result (4) along with Eq. (3) is based on the observation that the phase of the transition amplitudes $\langle r|F(K_l)|s\rangle \equiv F_{r,s}(K_l)$ between quasienergy eigenstates $|r\rangle$ and $|s\rangle$ depends strongly on the random detuning $k_l = K_l - K$ of the kick. In the presence of noise we set

$$\overline{F_{r,s}(K_l)} = \delta_{r,s} \exp[i\omega_r - 1/(2t_c)], \tag{8}$$

while in the absence of noise $F_{r,s}(K) = \delta_{r,s} \exp(i\omega_r)$, where ω_r is the quasienergy of the state.

The transition amplitudes $F_{r,s}$ enter the variance of the momentum $\mathrm{var}\, p(t) = \sum_{t',t''=0}^{t-1} C(t', t'')$ when the force-force correlation $C(t', t'') = \overline{\langle K_{t'} K_{t''} \sin\theta_{t'} \sin\theta_{t''}\rangle}$ is expanded in the basis of quasienergy eigenstates [20],

$$C(t', t'') = \sum_{\{r_n, s_n\}} \langle r_{t'}| \sin\theta |s_{t'}\rangle\langle s_{t''}| \sin\theta |r_{t''}\rangle \times \prod_{l,m=t''}^{t'-1} \overline{K_{t'} K_{t''} F^*_{r_{l+1},r_l}(K_l) F_{s_{m+1},s_m}(K_m)}. \tag{9}$$

In the sum (9), only those terms which satisfy either (a) $r_n \equiv r$ and $s_n \equiv s$ or (b) $r_n \equiv s_n$ survive the average in the random-phase approximation (8). The terms (b), however, have a zero contribution since they are multiplied by $\langle r| \sin\theta |r\rangle = 0$, a property which follows from the symmetry $(\theta, p) \rightarrow (-\theta, -p)$ of the kicked rotator. Each of the terms (a) further carries a weight factor $\exp[-N(t', t'')/t_c]$, which has to be averaged using Eq. (8). The averaging procedure over the renewal process then leads to the decoherence factor $\mathcal{D}(t', t'')$ defined in Eq. (3). Moreover, for $t' = t''$ the noise also gives rise to a contribution to the typical force acting on the particle of the form $\bar{K}^2_{t'}\langle \sin\theta^2\rangle = K^2/2 + (\kappa/2)f(t')$. Collecting all terms, this yields

$$C(t', t'') = C_0(t', t'')\mathcal{D}(t', t'') + \frac{\kappa}{2} f(t')\delta_{t',t''}, \tag{10}$$

where $C_0(t', t'')$ is the force-force correlation function in the absence of noise. Equation (4) then follows by converting sums into integrals and integrating by parts.

Conclusions.—Nonexponential relaxation is familiar from the physics of complex systems—such as disordered crystals and glassy materials—and occurs whenever there is no clear separation of time scales between system and environment [23]. At the quantum level, the absence of a scale separation manifests itself in a nonexponential loss of phase coherence. We have shown that such a behavior can be induced in atom-optical systems by properly engineering the environment. By applying Lévy noise with a controllable exponent α, the environment time scale—the mean waiting time $\bar{\tau}$ between noisy events—is finite for $\alpha > 1$ and can be made divergent by choosing $\alpha \leq 1$. For the quantum Lévy kicked rotator, the power-law decay of the decoherence factor and the nonexistence of a finite decoherence rate reveals itself in quantum subdiffusion of the momentum, a distinct signature that should be observable experimentally using cold atoms in pulsed optical fields. The high degree of control and tunability of atom-optical systems make the Lévy kicked rotator an ideal tool for the study of the crossover from fast to slow decoherence. Complex systems with algebraic relaxation are known to never attain equilibrium; the possibility of having noisy quantum systems that never become classical appears as a fascinating prospect.

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