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# MATCHING EXTERIOR TO INTERIOR SOLUTIONS IN WEYL GRAVITY: COMMENT ON "EXACT VACUUM SOLUTION TO CONFORMAL WEYL GRAVITY AND GALACTIC ROTATION CURVES"

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## ABSTRACT

Recently Mannheim and Kazanas presented the static, spherically symmetric exact vacuum solution of Weyl gravity and suggested that it could be fitted to general relativistic experimental tests in the solar system and to the observed galactic rotation curves. Here we discuss matching this solution to an interior one that satisfies the weak energy condition and a regularity condition at the center. We show that this leads to contradiction of Mannheim & Kazanas's suggestion.


*Subject headings:* cosmology: theory — gravitation — relativity

## 1. INTRODUCTION

Recently Mannheim & Kazanas (1989) have discussed the conformal invariant gravitational theory, so-called Weyl gravity, as a potential alternative to Einstein's general relativity. The theory is based on the local conformal invariance $[g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)]$ of geometry and the fourth-order gravitational action

$$I_w = -\alpha \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa}$$
$$\simeq -2\alpha \int d^4x(-g)^{1/2}[R_{\mu\nu}R^{\mu\nu} - (\tfrac{1}{3})(R^\nu_\nu)^2] , \qquad (1)$$

where $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor, $R_{\mu\nu}$ is the Ricci tensor, $R^\nu_\nu$ is the Ricci scalar, and $\alpha$ is a purely dimensionless coefficient. The action of equation (1), together with a matter Lagrangian $L_{\rm matt}$, leads to the following gravitational field equation:

$$-2\alpha[W^{(2)}_{\mu\nu} - \tfrac{1}{3}W^{(1)}_{\mu\nu}] = -2\alpha W_{\mu\nu} = -\tfrac{1}{2}T_{\mu\nu} , \qquad (2)$$

where $T_{\mu\nu} = \delta L_{\rm matt}/\delta g^{\mu\nu}$ is the energy-momentum tensor and $W^{(1)}_{\mu\nu}$ and $W^{(2)}_{\mu\nu}$ are expressed by

$$W^{(1)}_{\mu\nu} = 2g_{\mu\nu}(R^\alpha_\alpha)^{;\beta}{}_{;\beta} - 2(R^\alpha_\alpha)_{;\mu;\nu} - 2R^\alpha_\alpha R_{\mu\nu} + \tfrac{1}{2}g_{\mu\nu}(R^\alpha_\alpha)^2 , \qquad (3a)$$

$$W^{(2)}_{\mu\nu} = \tfrac{1}{2}g_{\mu\nu}(R^\alpha_\alpha)^{;\beta}{}_{;\beta} + R_{\mu\nu}{}^{;\sigma}{}_{;\sigma} - R_{\mu\sigma;\nu}{}^{;\sigma} - R_{\nu\sigma;\mu}{}^{;\sigma} - 2R^\sigma_\mu R_{\sigma\nu} + \tfrac{1}{2}g_{\mu\nu}R_{\sigma\tau}R^{\sigma\tau} . \qquad (3b)$$

"Weyl gravity" in the sense of Mannheim & Kazanas should not be confused with "Weyl geometry," where the parallel-transport does not preserve length. The above theory has been proposed first by Bach (1921), who took Weyl's idea of a conformally invariant gravitational theory and eliminated Weyl's additional vector field. Therefore, the tensor $W_{\mu\nu}$ of equation (2) is properly called the Bach tensor. Mannheim & Kazanas have published a series of papers to discuss Weyl gravity: a large set of exact solutions, these being the fourth-order analogs of the exterior Schwarzschild solution (Mannheim & Kazanas 1989); the Reissner-Nordstrøm, the Kerr, and the Kerr-Newman solutions (Mannheim & Kazanas 1991; Mannheim 1992a); the general structure of the field equation of the Weyl gravity (Kazanas & Mannheim 1991); conformal cosmology (in Weyl's gravity) with no cosmological constant (Mannheim 1990); and conformal gravity and the flatness problem (Mannheim 1992a). Furthermore, other authors have followed this direction (e.g., Wood & Nemiroff 1991). All these papers are based on Mannheim & Kazanas (1989) on the exact Schwarzschildlike solution.

In their 1989 paper, Mannheim & Kazanas showed that the static and spherically symmetric exact vacuum solution for Weyl gravity is given, up to a conformal factor, by the metric

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) , \qquad (4)$$

with

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2 , \qquad (5)$$

where $\beta$, $\gamma$, and $k$ are constants. Mannheim & Kazanas claimed that their solution could be fitted to all observational data, not only in the solar system but also on a galactic scale. As to the solar system, they considered $\gamma$ and $k$ to be sufficiently small. Then Weyl gravity "appears to enjoy the experimental successes of Einstein relativity," i.e., in their consideration $\beta$ would be similar to $GM$ in Schwarzschild solution. On a typical galactic scale, they took the $\gamma r$ term to be comparable in magnitude to the Newtonian potential $1/r$ term with typical galactic coefficient $10^{11}\, M_\odot G$. Then the $\gamma r$ term would cause a potential which grows with distance for $r > 10$ kpc and could be used to explain the observed galactic rotation curves without the need for dark matter. Finally, they considered $k$ to be the cosmological scalar curvature and the $kr^2$ term to be important at cosmological distances. Note that their theory allows a de Sitter spacetime as a vacuum solution without a cosmological constant. If all constants could be chosen at will, their theory would be very promising (Kazanas 1992). Unfortunately, this is not the case, as we will point out later.

It should be mentioned that a more general class of exact solutions of Bach theory (it is only an alternative name of Weyl gravity) has previously been discussed (Fiedler & Schimming

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1980). Mannheim & Kazanas's exact solution to the Reissner-Nordstrøm problem in Weyl gravity (Mannheim & Kazanas 1991), including the Schwarzschild problem as a special case of the Reissner-Nordstrøm problem, is just a subclass of Fiedler and Schimming's more general class of solutions given in their theorem 2.1 (Fiedler & Schimming 1980). Related material can also be found in a paper by Riegert (1984).

In our view, there are three serious problems with Mannheim & Kazanas's theory).

The first problem is related to the fact that the Bach tensor is trace-free. Therefore, the energy-momentum tensor is trace-free also. It is hard to see how normal matter with nonzero rest-mass (e.g., our Sun) can be modeled in such a theory. This problem is also related to a vacuum solution because its boundary conditions have to connect to an interior solution (see the third problem below).

The second problem is also related to the conformal invariance of the theory. Since the left-hand side of Bach's field equation (2) transforms homogeneously under a conformal transformation, the metric (4), as well as any other vacuum solution, remains a solution if it is multiplied by a conformal factor. Therefore, the timelike geodesics are not fixed (whereas the null geodesics are), and we need some additional information to determine the real orbit of motion of a massive test particle.

The third problem is the most important one. This is the main subject of our paper. As we know, any physical solution of differential equations not only satisfies the equations themselves but also satisfies certain boundary conditions. The above-mentioned constants $\beta$, $\gamma$, and $k$ (Mannheim & Kazanas 1989) should be related to boundary conditions, which are determined by an interior solution and by the behavior at infinity. So they are not completely free to be fitted to observational data, such as rotating velocities in a spiral galaxy. To be sure, Mannheim & Kazanas are well aware of this problem. Mannheim already discussed the relation of their exterior solution to an interior one and expressed the constant $\beta$ and $\gamma$ in terms of integrals over the source (Mannheim 1992b) (eqs. [5.9] and [5.10]). However, his result is not correct in general, since his equation (5.6) does not give the general solution to the differential equation $(rB)'''' = rf$. An arbitrary solution of the homogeneous equation, i.e., $B(r) = C_1 + (C_2/r) + C_3 r + C_4 r^2$, may also be added. The same mistake occurs in a recent follow-up paper by Mannheim & Kazanas (1994).

In this paper, we will discuss matching the exterior and interior solutions. To keep the argument as general as possible, we do not assume any specific form of $T^{\mu\nu}$ for the interior solution. We only assume the weak energy condition to be satisfied. Moreover, we assume the interior solution to admit a regular center. In such a general way, the constants in the exterior solution can be limited in a special region, which contradicts Mannheim & Kazanas's conclusion. Either the second term in equation (5) has a wrong sign or the $\gamma r$ term is larger in magnitude than the second one in the whole domain of the exterior solution. Then in any region (including the scale of solar system), Weyl gravity cannot "appear to enjoy the experimental successes of Einstein relativity"; i.e., in the solar system it does not agree with the four famous classic tests. Therefore, the whole series of papers by Mannheim & Kazanas has lost its physical basis.

In § 2, we rewrite the field equation (in Weyl gravity) of a static spherically symmetric metric in a slightly simpler way than Mannheim & Kazanas did. The weak energy condition is discussed in § 3, whereas § 4 is devoted to the regularity condition at the center. In § 5, the matching between interior and exterior solutions is made. Finally, in § 6 some conclusions are given.

## 2. THE FIELD EQUATION FOR SPHERICALLY SYMMETRIC AND STATIC METRICS

To calculate the field equation (2) for a spherically symmetric and static metric

$$ds^2 = -b(\rho)dt^2 + a(\rho)d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \quad (6)$$

we follow, by and large, Mannheim & Kazanas (1989). At first, we make a transformation of the radial coordinate, $\rho = p(r)$, where $p$ satisfies the differential equation $r^4 a[p(r)]b[p(r)]p'(r)^2 = p(r)^4$. This puts the metric (6) into the form

$$ds^2 = \frac{p(r)^2}{r^2}\left[-B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \quad (7)$$

where $B(r) = r^2 b[p(r)]/p(r)^2$. Since the Bach tensor $W_{\mu\nu}$ transforms homogeneously under a conformal transformation ($g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$, $W_{\mu\nu} \mapsto \Omega^{-2} W_{\mu\nu}$), it is sufficient to calculate $W_{\mu\nu}$ for metrics of the form of equation (7) with $p(r) = r$. This is still rather tedious but straightforward. If we substitute

$$u = \frac{1}{r}\,, \quad h(u) = u^2 B\left(\frac{1}{u}\right), \quad (8)$$

we find for the nonvanishing contravariant components of the Bach tensor

$$W^{rr} = \frac{hu^2}{6}\left(h'h''' - \frac{h''^2}{2} + 2\right), \quad (9a)$$

$$W^{tt} = \frac{u^6}{6h}\left(-2hh'''' - h'h''' + \frac{h''^2}{2} - 2\right), \quad (9b)$$

$$W^{\theta\theta} = \sin^2\theta\, W^{\phi\phi} = \frac{u^6}{6}\left(-hh'''' - h'h''' + \frac{h''^2}{2} - 2\right). \quad (9c)$$

Equations (9a) and (9b) imply

$$h'''' = \mu\,, \quad (10)$$

with

$$\mu(u) \equiv -\frac{3}{u^2}\left[\frac{W^{rr}(u)}{h(u)^2} + \frac{W^{tt}(u)}{u^4}\right]. \quad (11)$$

Equation (10) will be the most essential tool for our reasoning. In later parts of this paper, we evaluate this equation for an interior solution with regular center whose energy-momentum tensor satisfies the weak energy condition. Here we briefly rederive the vacuum solution. In the vacuum case, $\mu$ vanishes, and equation (10) says that $h$ is a third-degree polynomial in $u$. However, the coefficients of this polynomial are not completely independent, since equation (9) must be satisfied with $W^{\mu\nu} = 0$. This shows that $h(u)$ must be of the form

$$h(u) = -\beta(2 - 3\beta\gamma)u^3 + (1 - 3\beta\gamma)u^2 + \gamma u - k\,, \quad (12)$$

where $\beta$, $\gamma$, and $k$ are arbitrary constants. Upon resubstituting according to equation (8), this coincides with the result of Mannheim & Kazanas (1989); see equation (5) above.

## 3. THE WEAK ENERGY CONDITION

Recall that an energy-momentum tensor $T^{\mu\nu}$ is said to satisfy the weak energy condition if and only if $V_\mu V^\mu < 0$ implies $T^{\mu\nu} V_\mu V_\nu \geq 0$ (Hawking & Ellis 1973, p. 89). In physical terms, this means that the energy density is nonnegative for any observer. Since the Bach tensor and, thus, the energy-momentum tensor transforms homogeneously under a conformal transformation, the weak energy condition is conformally invariant. Hence, we restrict our considerations to a metric (7) with $p(r) = r$. In such a metric, the vector field

$$V^\mu = \sqrt{\frac{1}{B} + \frac{K^2}{B^2}}\, \delta^\mu_t + K\delta^\mu_r$$

satisfies $V^\mu V_\mu = -1$ for any $K$. Thus, the weak energy condition implies

$$BT^{tt}\left(1 + \frac{K^2}{B}\right) + T^{rr}\,\frac{K^2}{B^2} \geq 0 \quad \text{for all } K\,.$$

Dividing by $K^2$ and letting $K \to \infty$, we find

$$T^{tt} + \frac{1}{B^2}\, T^{rr} \geq 0\,,$$

and, by equations (2), (8), and (11),

$$\alpha\mu(u) \leq 0\,. \tag{13}$$

This inequality, which is a consequence of the weak energy condition, is the main result of this section. It says that the sign of the function $\mu$ is determined by the sign of the "gravitational constant" $\alpha$. Please note that it is not clear, at the outset, whether $\alpha$ is positive or negative.

Although our reasoning will be based on the weak energy condition without specifying $T^{\mu\nu}$ any further, it might be helpful to mention an example. Let us consider a perfect fluid with the extreme equation of state $P = \rho/3$ (any other equation of state contradicts the condition that $T^{\mu\nu}$ is trace-free):

$$T^{\mu\nu} = \frac{4}{3}\,\rho U^\mu U^\nu + \frac{\rho}{3}\, g^{\mu\nu}\,.$$

Here the four-velocity $U^\mu$ satisfies $U_\mu U^\mu = -1$, and the weak energy condition is satisfied if and only if the mass density $\rho$ is nonnegative. Let us restrict ourselves to the case in which the fluid is at rest in our static metric (7) with $p(r) = r$, i.e., $U^\mu = (1/B^{1/2})\delta^\mu_t = (u/h^{1/2})\delta^\mu_t$. Then the field equation (2) yields

$$W^{rr} = \frac{\rho h}{12\alpha u^2}\,,$$

$$W^{tt} = \frac{\rho u^2}{4\alpha h}\,.$$

After an elementary calculation using equations (9a) and (9b), we obtain

$$\frac{\rho h^2}{u^4} = \tilde{K} = \text{const}\,. \tag{14}$$

This shows that a perfect fluid solution cannot be joined as an interior solution to an exterior vacuum solution. The reason is that the junction conditions, to be derived in § 5, require $h$ and $W^{rr}$ to be continuous, which implies that $\rho$ must be continuous. Thus, $\rho$ must go to zero for $r \to r_0$ (i.e., $u \to 1/r_0$) if we want to join a vacuum solution there. This, however, is impossible if equation (14) holds with $\tilde{K} \neq 0$. This interesting result can be rephrased in the following way. In Bach's theory, a (static and spherically symmetric) perfect fluid ball must extend to infinity.

This example will not be used later on. It was included only for the sake of illustration.

## 4. REGULARITY AT THE CENTER

A reasonable interior solution should be regular at the center of symmetry in the sense that there is no true singularity. (A "true" singularity is a singularity that cannot be removed by a coordinate transformation.) It is our aim to exploit this regularity condition in a conformally invariant way. Therefore, it is not sufficient to use the standard results on regularity (Kramer et al. 1980, p. 192), and we give a detailed derivation in the following. For our metric (7), the center of symmetry is characterized by $p(r) = 0$, as can be read from the factor preceding $d\theta^2 + \sin^2\theta\, d\phi^2$. Clearly, the metric (7) becomes singular at $p(r) = 0$ in the coordinates used. We now require that this is a mere coordinate singularity. Owing to spherical symmetry, this means that the metric becomes manifestly regular at $p(r) = 0$ if the radial coordinate is appropriately rescaled and if the usual coordinate singularity of polar coordinates is removed by transformation to, e.g., Cartesian coordinates. It is our aim to show that, on this assumption, $p(r) \to 0$ is equivalent to $r \to 0$ and that $B(r) \to 1$ for $r \to 0$. To that end, we restrict our considerations to the plane $\theta = \pi/2$ and introduce new coordinates

$$\begin{aligned} x &= q(r)\cos\phi\,, \\ y &= q(r)\sin\phi\,, \end{aligned} \tag{15}$$

with a differentiable function $q(r)$ unspecified so far. This puts the metric (7), restricted to $\theta = \pi/2$, into the form

$$\begin{aligned} ds^2 = &-\frac{p^2 B}{r^2}\,dt^2 + \left[\frac{p^2}{q^2} + \cos^2\phi\left(\frac{p^2}{q'^2 r^2 B} - \frac{p^2}{q^2}\right)\right]dx^2 \\ &+ \left[\frac{p^2}{q^2} + \sin^2\phi\left(\frac{p^2}{q'^2 r^2 B} - \frac{p^2}{q^2}\right)\right]dy^2 \\ &+ 2\sin\phi\cos\phi\left(\frac{p^2}{q'^2 r^2 B} - \frac{p^2}{q^2}\right)dx\,dy\,. \end{aligned} \tag{16}$$

Our regularity assumption guarantees that $q(r)$ can be chosen in such a way that the metric (16) is manifestly regular at $p(r) = 0$. This means that each metric coefficient in equation (16) has a well-defined limit for $p(r) \to 0$ which is independent of direction $\phi$, and that in this limit equation (16) still gives a metric of signature $(-+++)$. From $g_{tt}$ we read that this requires

$$\lim_{p(r)\to 0} \frac{p(r)^2 B(r)}{r^2} = C^2 > 0\,, \tag{17}$$

where $C^2 > 0$ is due to the signature not to be changed. From $g_{xx}$, $g_{yy}$, and $g_{xy}$ we find, on the other hand,

$$\lim_{p(r)\to 0}\left[\frac{p(r)^2}{q'(r)^2 r^2 B(r)} - \frac{p(r)^2}{q(r)^2}\right] = 0\,, \tag{18}$$

and

$$\lim_{p(r)\to 0} \frac{p(r)^2}{q(r)^2} = A^2 > 0 , \tag{19}$$

where $A^2 > 0$ is again due to the signature not to be changed. Here we used the fact that the limit should not depend on direction. Without loss of generality, we assume $q'(r) > 0$ (and $A$ and $C > 0$). The equations (17), (18), and (19) imply

$$\frac{q'(r)r^2}{q(r)^2} \to \frac{A}{C} , \tag{20}$$

for $p(r) \to 0$. Hence, for some region $0 < p(r) < \epsilon$, we have

$$\frac{A}{2Cr^2} < \frac{q'(r)}{q(r)^2} < \frac{2A}{Cr^2} . \tag{21}$$

Upon integration, this yields

$$\frac{A}{2C}\left(\frac{1}{r} - \frac{1}{r_0}\right) < \frac{1}{q(r)} - \frac{1}{q(r_0)} < \frac{2A}{C}\left(\frac{1}{r} - \frac{1}{r_0}\right) . \tag{22}$$

Here $r_0$ is an appropriately chosen integration constant and equation (22) holds for $r < r_0$. This shows that the conditions $q(r) \to 0$ and $r \to 0$ are equivalent. On the other hand, by equation (19), $p(r) \to 0$ and $q(r) \to 0$ are equivalent. So we have proved our first claim, i.e., the equivalence of $p(r) \to 0$ and $r \to 0$. To prove our second claim, i.e., $B(r) \to 1$, we observe that, by the Bernoulli-l'Hôspital rule, $q(r)/r$ and $q'(r)$ have the same limit. Hence, by equation (20),

$$\frac{q(r)}{r} \to \frac{C}{A} , \tag{23}$$

for $r \to 0$. In combination with equations (17) and (19), this proves

$$B(r) \to 1 , \tag{24}$$

for $r \to 0$. The crucial point is that equation (24) gives a necessary condition for regularity at the center that does not involve the function $p$. In terms of the function $h(u)$ introduced in equation (8), equation (24) takes the form

$$\frac{h(u)}{u^2} \to 1 , \tag{25}$$

for $u \to \infty$. Successive application of the Bernoulli-l'Hôspital rule shows

$$\frac{h'(u)}{u} \to 2 , \tag{26}$$

$$h''(u) \to 2 , \tag{27}$$

$$uh'''(u) \to 0 , \tag{28}$$

for $u \to \infty$. The four conditions (25)–(28), necessary for regularity at the center, will be made use of in the next section.

## 5. MATCHING OF EXTERIOR AND INTERIOR SOLUTION

Here we want to join an exterior vacuum solution to an interior solution. The interior solution is supposed to satisfy the weak energy condition and to be regular at the center, but it is not specified any further. We allow for discontinuities of the energy-momentum tensor at the boundary between interior and exterior solution, but we assume that the energy-momentum tensor has no Dirac-delta singularities. To put this another way, we exclude surface energy densities. (These are the usual assumptions to derive junction conditions between interior and exterior solutions.) More specifically, we agree upon the following stipulation. We assume that we are given a metric (7) on the domain $0 < r < \infty$, that satisfies the following conditions:

i) Bach's field equation (2) is satisfied everywhere with an energy-momentum tensor that has no Dirac-delta singularities. However, it may have discontinuities. (In a mathematically rigorous setting, this requires viewing field eq. [2] in a distributional sense.)
ii) For $r > r_0$, the vacuum Bach equation is satisfied.
iii) For $r < r_0$, the weak energy condition is satisfied.
iv) For $r \to 0$, $B(r) \to 1$.

Our aim is to show that there is no region in the entire domain $r_0 < r < \infty$ in which the metric is close to the standard Schwarzschild metric (or to a metric that is conformally equivalent to the Schwarzschild metric). Consequently, a metric that satisfies conditions (i)–(iv) above cannot be a viable model of our solar system.

To prove this proposition, we proceed in the following way. Given a metric (7) that satisfies conditions (i)–(iv) above, we can find, by an appropriate conformal transformation, a metric of the form of equation (7) with $p(r) = r$ that also satisfies conditions (i)–(iv). This new metric is completely determined by the function $B(r)$ or, if we use the substitution equation (8), by the function $h(u)$. Condition (i) guarantees that $h(u)$ is continuous everywhere. (At any hypothetical discontinuity of $h$ the right-hand sides of eqs. [9a] and [9b] were not well defined. Thus, Bach's field equation could not be satisfied there, not even in a distributional sense.) Hence, the function $\mu$ defined in equation (11) has at most discontinuities, but no delta singularities. So equation (10) guarantees that

$$h''' \text{ is continuous} \tag{29}$$

everywhere. (Otherwise its derivative $h''''$ had a delta singularity.) Repeating this argument, we find that

$$h'' \text{ is continuous} \tag{30}$$

everywhere. (Otherwise $h'''$ had a delta singularity in contradiction to statement [29].) By the same token, $h'$ and $h$ must be continuous everywhere. Together with equation (9), this shows that $W^{rr}$ is continuous everywhere, whereas $W^{tt}$, $W^{\theta\theta}$, and $W^{\phi\phi}$ and, thus, $\mu$ may have discontinuities. Next we exploit condition (iv), which implies equations (25)–(28). We restrict our consideration to the interior region $u > 1/r_0$. Integration of equation (10) together with equation (28) yields

$$h'''(u) = -\int_u^\infty \mu(\tilde{u})d\tilde{u} . \tag{31}$$

Integrating once more and using equation (22) results in

$$h''(u) = 2 + \int_u^\infty \int_{\hat{u}}^\infty \mu(\tilde{u})d\tilde{u}\, d\hat{u} . \tag{32}$$

On the other hand, condition (ii) says that in the exterior region $h(u)$ is given by the vacuum solution (12). With statements (29) and (30), we can match this exterior solution at

$u = 1/r_0$ to the interior solution in terms of equations (31) and (32). This results in

$$M = 6\beta(2 - 3\beta\gamma) , \tag{33}$$

$$2 + Z = 2(1 - 3\beta\gamma) - \frac{6\beta}{r_0}(2 - 3\beta\gamma) , \tag{34}$$

with

$$M \equiv \int_{1/r_0}^{\infty} \mu(u)du , \tag{35}$$

and

$$Z \equiv \int_{1/r_0}^{\infty}\int_{u}^{\infty} \mu(\tilde{u})d\tilde{u}\, du . \tag{36}$$

Finally, we exploit the weak energy condition (iii). This makes it necessary to distinguish two cases.

*Case 1*: $\alpha > 0$. In this case, by inequality (13), we have $\mu \leq 0$. Hence equation (33) implies $\beta(2 - 3\beta\gamma) \leq 0$. This means that the term proportional to $1/r$ in our exterior vacuum solution has a positive sign (such as in the standard Schwarzschild solution with *negative* mass). Therefore it is unphysical.

*Case 2*: $\alpha < 0$. Now inequality (13) implies $\mu \geq 0$. From equations (33) and (34), we find

$$\beta = \frac{M}{(3M/r_0) + 12 + 3Z} \geq 0 , \tag{37}$$

$$-\gamma = \frac{1}{6\beta}\left(\frac{M}{r_0} + Z\right) \geq 0 . \tag{38}$$

By equation (37), $1/\beta \geq 3/r_0$, such that equation (38) implies $-\gamma \geq M/2r_0^2$. Together with equation (33), this gives the following sequence of inequalities:

$$|\gamma| = -\gamma \geq \frac{M}{2r_0^2} = \frac{3\beta}{r_0^2}(2 - 3\beta\gamma) \geq \frac{\beta}{r_0^2}(2 - 3\beta\gamma) \quad (\geq 0) . \tag{39}$$

Hence, from equation (39) we find for all $r \geq r_0$

$$|\gamma r| \geq \left|\frac{\beta}{r}(2 - 3\beta\gamma)\right| . \tag{40}$$

Equation (40) means that nowhere on the domain of the exterior solution can the term proportional to $r$ be neglected in comparison to the term proportional to $1/r$.

To recapitulate, the assumptions stated in the beginning of this section lead to the following conclusion: In any case, the exterior vacuum solution is totally different from the standard Schwarzschild solution everywhere and would not fit any experimental test in the solar system.

As an aside, we briefly point out that the same conclusion follows from the equations

$$\beta(2 - 3\beta\gamma) = \tfrac{1}{6}\int_0^{r_0} f(r')r'^4\, dr' \tag{41}$$

and

$$\gamma = -\tfrac{1}{2}\int_0^{r_0} f(r')r'^2\, dr' , \tag{42}$$

given by Mannheim (1992b) as his equations (5.9) and (5.10), where $f$ is related to our $\mu$ by $f(r) = \mu(u)u^6$. (Whereas eq. [41] is equivalent to our eq. [33], eq. [42] is in general not true, as already mentioned in the introduction.) From equation (41), we find in case 1 that $\beta(2 - 3\beta\gamma) \leq 0$, whereas in case 2 we find from equations (41) and (42) for $r \geq r_0$

$$-\gamma = \frac{1}{2}\int_0^{r_0} f(r')\frac{r'^4}{r'^2}\, dr' \geq \frac{1}{2}\int_0^{r_0} f(r')\frac{r'^4}{r^2}\, dr' = \frac{3\beta}{r^2}(2 - 3\beta\gamma) , \tag{43}$$

which again implies equation (40).

## 6. CONCLUSIONS

In our view, the argument presented in this paper clearly shows that Weyl gravity (= Bach theory) is not able to give a viable model of our solar system. The only way to be in agreement with observation is to assume that either the energy density is negative somewhere inside the Sun or there is a singularity at its center. We are of the opinion that both assumptions are highly unphysical and should be rejected. As a result, then, all the follow-up papers by Mannheim & Kazanas mentioned in the introduction have lost their basis, and it seems as if, unfortunately, Bach theory has no prospect of physical success at all.

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