On Spacetime Models with an Isotropic Hubble Law

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Abstract

We consider a Lorentzian manifold $(\mathcal{M}, g)$ with an observer field (timelike vector field) V. Along each lightlike geodesic the redshift z and the angular diameter distance D are then well-defined functions. (Instead of the angular diameter distance one may equivalently use the corrected or uncorrected luminosity distance.) $(\mathcal{M}, g, V)$ is said to admit an isotropic Hubble law at $p \in \mathcal{M}$ if all past-oriented lightlike geodesics issuing from p yield the same z-D-relation. For infinitesimally short lightlike geodesics (i.e., to within linear approximation with respect to the variable D) it is well known that an isotropic Hubble law holds at all points $p \in \mathcal{M}$ if and only if V is freely falling and shear-free. Here we derive the necessary and sufficient conditions for an isotropic Hubble law beyond the linear regime. To that end we expand the z-D-relation in a Taylor series (Kristian-Sachs series) and we investigate the validity of an isotropic Hubble law order by order. In particular, we prove that $(\mathcal{M}, g, V)$ admits an isotropic Hubble law of third order at every point $p \in \mathcal{M}$ if and only if $(\mathcal{M}, g, V)$ is either redshift-free or a Robertson-Walker model (locally around every point $p \in \mathcal{M}$).

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1 Introduction

In the year 1929 Edwin Hubble postulated his famous law according to which there is a linear and isotropic relation between the redshift z and the distance D of galaxies. Here "linear" means that the relation is of the form $z = H_1 D$ and "isotropic" means that the relation is independent of the direction, i.e., that the coefficient H_1 does not vary over the sky. If such a law holds for an observer at our position in the universe, it is reasonable to assume that it also holds for an observer at any other position in the universe, with a "Hubble constant" H_1 that may depend on the position.

Theoretically, an isotropic relation between z and D can be proven to hold in Robertson-Walker models. However, this relation is not linear except for very special Robertson-Walker models. For a systematic study of the z-D-relation in arbitrary world models it is therefore reasonable to view the linear Hubble law as the leading term in a series expansion (following a pioneering paper by Kristian and Sachs [7])

$$z = H_1 D + \cdots + H_n D^n + O(D^{n+1}) \tag{1}$$

and to investigate the isotropy for the coefficients H_1, H_2, etc. separately.

This is the program we want to carry through in this paper. To that end we proceed in the following way. We consider arbitrary kinematical world models, given by a Lorentzian manifold $(\mathcal{M}, g)$ and a timelike vector field V. Along each lightlike geodesic the redshift z and the distance D are then well-defined functions of the curve parameter. More precisely, there are several different measures for the distance and we have to decide which one we want to use. The most relevant measures for distance to be used in mathematical cosmology are the so-called *angular diameter distance* D, the so-called *luminosity distance* $\hat{D}$ and the so-called *corrected luminosity distance* $\tilde{D}$. We prefer to work with D but we also show that all results can be equivalently reformulated in terms of $\hat{D}$ or $\tilde{D}$. In practical terms, D is the relevant quantity if distance is measured by comparing the apparent size of an object with its actual size, whereas $\hat{D}$ and $\tilde{D}$ are the relevant quantities if distance is measured by comparing the apparent brightness of an object with its actual brightness.

In our arbitrary world model we consider the relation between z and D along each lightlike geodesic. We want to characterize the situation that this relation is the same for all lightlike geodesics issuing into

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the past from a common initial point $p \in \mathcal{M}$. Writing the z-D-relation in terms of a series expansion as in (1), we say that $(\mathcal{M}, g, V)$ admits an *isotropic Hubble law of* n^{th} *order* at p if for $i \leq n$ the coefficient H_i takes the same value for all lightlike geodesics issuing into the past from p. We call $(\mathcal{M}, g, V)$ an n^{th} *order Hubble model* if this is true for all points $p \in \mathcal{M}$.

The paper is organized as follows. In Section 2 we specify our terminology and we recall some basic facts about redshifts and distances in arbitrary kinematical world models. Section 3 is devoted to first order Hubble models. In particular, we rederive the known result (cf. Ehlers [5]) that $(\mathcal{M}, g, V)$ is a first order Hubble model if and only if V is shear-free and geodesic. In Section 4 we discuss second order Hubble models and we demonstrate, by way of example, that a second order Hubble model may have a non-vanishing rotation and a non-vanishing expansion. In Section 5 we show that a third order Hubble model is automatically an n^{th} order Hubble model for arbitrarily large n, i.e., that with studying the third order case our analysis has come to a natural end. In particular, we show that a third order Hubble model is either redshift-free or a Robertson-Walker model (locally around any one point).

According to this result, a 4-dimensional Lorentzian manifold $(\mathcal{M}, g)$ is a Robertson-Walker spacetime if it admits an observer field V such that for any point $p \in \mathcal{M}$ the z-D-relation is isotropic up to third order and H_1 is different from zero. This shows that, in principle, a family of observers can find out whether they are living in a Robertson-Walker spacetime just by examining the z-D-relation up to third order terms. The reader is encouraged to compare this result with other characterizations of Robertson-Walker spacetimes, see, e.g., Karcher [16] or Koch-Sen [18]. Also, it might be helpful to compare our results with the work of Maartens and Matravers [27] on isotropy of observations in cosmology. One of the main differences is in the fact that the latter restrict to dust solutions of Einstein's field equation from the outset.

2 Distance-redshift-relation in arbitrary kinematical world models

We consider kinematical world models in the sense of the following definition.

Definition 2.1. $(\mathcal{M}, g, V)$ is called a *kinematical world model* iff

(a) $\mathcal{M}$ is a connected 4-dimensional real C^∞ manifold whose topology satisfies the axiom of second countability and the Hausdorff axiom;

(b) g is a C^∞ Lorentzian metric on $\mathcal{M}$, i.e., a covariant symmetric second rank C^∞ tensor field of signature $(+,+,+,-)$;

(c) V is a C^∞ vector field on $\mathcal{M}$ with $g(V,V) = -1$.

In applications to cosmology we interpret the integral curves of V as the world lines of observers following the mean flow of luminous matter in the universe. The normalization condition $g(V,V) = -1$ requires those integral curves to be parametrized by proper time. We speak of a "kinematical" world model to emphasize that V need not satisfy any equation of motion and that Einstein's field equation is not considered.

According to the rules of general relativistic kinematics, lightlike geodesics in a Lorentzian spacetime model $(\mathcal{M}, g)$ are to be interpreted as light rays. (Throughout this paper we restrict our considerations to light rays which are not directly influenced by matter.) In a kinematical world model $(\mathcal{M}, g, V)$ we have, in addition, a distinguished observer field V. This allows us to introduce, along each light ray, the notions of redshift, angular diameter distance and corrected luminosity distance. In the following we recall the definitions of these notions and some of their basic properties.

For our purposes it will be convenient to fix a particular past-oriented parametrization along each light ray. Therefore we introduce the following notation. If $(\mathcal{M}, g, V)$ is a kinematical world model, we denote by $\mathcal{L}(\mathcal{M}, g, V)$ the set of all curves $\lambda : [0, s_0] \longrightarrow \mathcal{M}$ whose tangent field $K = \lambda' : [0, s_0] \longrightarrow T\mathcal{M}$ satisfies the conditions

$$\nabla_K K = 0 \quad \text{and} \quad g\big(K, K\big) = 0\,, \tag{2}$$

$$g\big(V_{\lambda(0)}, K(0)\big) = 1\,. \tag{3}$$

Here and in the following ∇ denotes the Levi-Civita connection of g. The length of the parameter interval is not specified, i.e., s_0 is a positive value depending on the individual curve.

2.1 Redshift

Along each curve $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$, the redshift is given by the formula

$$\frac{\nu(s_2)}{\nu(s_1)} = \frac{g\big(V_{\lambda(s_2)}, K(s_2)\big)}{g\big(V_{\lambda(s_1)}, K(s_1)\big)} \tag{4}$$

for all s_1 and s_2 in $[0, s_0]$, where $\nu(s)$ denotes the frequency measured by the V-observer at $\lambda(s)$. The well-known formula (4) is usually attributed to Schrödinger [4] although it can be traced back to a 1932 paper by Kermack, McCrea and Whittaker [2]. For a simple and elegant proof of this formula we refer to Brill [9]. In the particle picture, where the frequency is interpreted as the energy of a photon, (4) is quite evident without any calculation.

As usual in cosmology, we express the redshift in terms of a function $z : [0, s_0] \longrightarrow \mathbb{R}$ which is assigned to each $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ by

$$z(s) = \frac{\nu(s)}{\nu(0)} - 1\,. \tag{5}$$

Owing to our initial condition (3), we can then read from the general redshift formula (4) that z takes the form

$$z(s) = g\big(V_{\lambda(s)}, K(s)\big) - 1\,. \tag{6}$$

The function z satisfies, of course, the initial condition $z(0) = 0$. In "reasonable" cosmological models z is supposed to be strictly increasing for each $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$. In an arbitrary kinematical world model, however, z need not be a monotonous function of the parameter s. Please note that a negative value of $z(s)$ indicates that the V-observer at $\lambda(0)$ sees the V-observer at $\lambda(s)$ blueshifted.

If $(\mathcal{M}, g, V)$ is conformally stationary, i.e., if V is proportional to a conformal Killing vector field, the redshift can be calculated in a particularly simple way, according to the following theorem.

Theorem 2.2. *Let $(\mathcal{M}, g, V)$ be a kinematical world model and $f : \mathcal{M} \longrightarrow \mathbb{R}$ a C^∞ function. Then the following two properties are equivalent.*

(a) The vector field $W = e^f V$ is a conformal Killing vector field, i.e., the Lie derivative $L_W g$ is a multiple of g.

(b) f is a redshift potential *for $(\mathcal{M}, g, V)$, i.e., along every curve $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ the redshift z is given by $\ln\big(1 + z(s)\big) = f\big(\lambda(0)\big) - f\big(\lambda(s)\big)$, where $\ln$ means natural logarithm.*

Here the name "redshift potential" for f indicates that the redshift is determined by the difference of the values of f at the points of reception and emission. For a proof of Theorem 2.2 we refer to Hasse and Perlick [22]. Spacetimes admitting a redshift potential have also been investigated by Dautcourt [21]. The criterion given in the latter reference for a spacetime to admit a redshift potential is easily verified to be equivalent to conformal stationarity.

It is obvious that a redshift potential, if it exists, is unique up to an additive constant. We now specify Theorem 2.2 to the case $f = 0$. Since a conformal Killing vector field V that satisfies the normalization condition $g(V, V) = -1$ must be a Killing vector field, this immediately yields the following corollary.

Theorem 2.3. *A kinematical world model $(\mathcal{M}, g, V)$ is redshift-free (i.e., the redshift z is identically zero for every $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$) if and only if V is a Killing vector field (i.e., $L_V g = 0$).*

Lorentzian manifolds admitting a Killing vector field normalized to -1 are sometimes called *ultrastationary*. Thus, ultrastationarity of $(\mathcal{M}, g)$ is necessary and sufficient for the existence of an observer field V such that all V-observers see each other without redshift.

2.2 Angular diameter distance

In a kinematical world model $(\mathcal{M}, g, V)$ there are several different ways to define the distance between source and receiver of a light ray. In this subsection we consider the so-called *angular diameter distance* which is based on the intuitive idea that the farther an object is away the smaller it looks. More precisely, for every light ray $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ the angular diameter distance is defined as a function $D : [0, s_0] \longrightarrow \mathbb{R}$ in the following way. Around the central ray λ we consider an infinitesimally thin bundle of neighboring rays issuing from the same point $\lambda(0)$. ("Infinitesimally thin" means that the neighboring rays are to be modeled as Jacobi fields along λ, not as curves in $\mathcal{M}$.) For each parameter value s, we can then consider the ratio of the cross-sectional area of the bundle at the point $\lambda(s)$ and the opening angle (solid angle) of the bundle at $\lambda(0)$, measured in the rest space of the respective V-observer. The square-root of this ratio is, by definition, the angular diameter distance $D(s)$. It can be shown (see, e.g., Seitz, Schneider and Ehlers [26]) that the function D satisfies the differential equation

$$D''(s) = -\Big(\tfrac{1}{2}\mathrm{Ric}\big(K(s), K(s)\big) + |\hat{\sigma}(s)|^2\Big) D(s) \,, \tag{7}$$

and the initial conditions

$$D(0) = 0 \quad \text{and} \quad D'(0) = 1 \,. \tag{8}$$

Here Ric denotes the Ricci tensor of the metric g and $\hat{\sigma}$ denotes the (complex) shear of the infinitesimal bundle.

The initial conditions (8) guarantee that, for small parameter values s, the angular diameter distance is monotonically increasing along λ, as a "distance" is supposed to be. This implies, in particular, that the function $s \longmapsto D(s)$ is invertible. Farther away from $s = 0$, however, D might be decreasing, thereby indicating a refocusing of the bundle.

The differential equation (7) is often called the *focusing equation*. If we want to determine the function D with the help of (7) and (8), we have, of course, to know the Ricci tensor Ric and the shear $\hat{\sigma}$. If the metric is given, it is a straightforward (though tedious) matter to calculate Ric. To determine the shear, one has to solve the well-known first order system of differential equations for the optical scalars (shear, rotation and expansion), first established by Sachs [6], with initial conditions determined by the fact that the bundle has a vertex at $s = 0$. We do not want to go into an analysis of these equations here. We just mention the following two facts.

(A) If $(\mathcal{M}, g)$ is conformally flat, $\hat{\sigma}$ is identically zero. (For a proof we refer to Sachs [6].)

(B) The shear $\hat{\sigma}$ has a singularity at all points where the cross-section of the bundle collapses into a line, i.e., at all non-degenerate conjugate points. $\hat{\sigma}$ has a zero at all points where the cross-section of the bundle collapses into a point, i.e., at the vertex $s = 0$ and at all degenerate conjugate points. (For a proof we refer to Seitz, Schneider and Ehlers [26].)

If the interval $[0, s_0]$ has been chosen small enough such that it does not contain a non-degenerate conjugate point, the linear differential equation (7) and the initial values (8) determine a unique solution $D : [0, s_0] \longrightarrow \mathbb{R}$. As mentioned above, this function D can be inverted (after shortening the interval $[0, s_0]$ if necessary), i.e., the parameter s can be viewed as a function of $D(s)$. Upon inserting the resulting expression into (6) we get the redshift $z(s)$ as a function of the angular diameter distance $D(s)$, i.e., we get the z-D-relation for the kinematical world model under consideration.

Only in very special world models is it possible to do all these calculations explicitly and to get the z-D-relation in closed form. Usually one resorts to series expansions (cf. Kristian and Sachs [7]), i.e., one is satisfied with getting the z-D-relation in the form

$$z(s) = H_1\, D(s) \; + H_2\, D(s)^2 \; + \; \dots + H_n\, D(s)^n \; + \; O\big(D(s)^{(n+1)}\big) \tag{9}$$

for some $n \in \mathbb{N}$. Please note that, owing to our C^∞ assumption on the metric and on the observer field, Taylor's theorem guarantees that such a representation is valid for all $n \in \mathbb{N}$. Thus, to each light ray $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ there is assigned an infinite sequence of well-defined coefficients H_1, H_2, etc. To characterize the situation that these coefficients are the same for all light rays issuing from a particular point into the past, we introduce the following terminology.

Definition 2.4. The kinematical world model $(\mathcal{M}, g, V)$ is said to admit an *isotropic Hubble law of n^{th} order* at p if for $i \leq n$ the coefficient H_i takes the same value for all lightlike geodesics $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ with $\lambda(0) = p$. $(\mathcal{M}, g, V)$ is called an n^{th} *order Hubble model* if it admits an isotropic Hubble law of n^{th} order at all points $p \in \mathcal{M}$.

To calculate the coefficients $H_1, \dots, H_n$ for a ray $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ one proceeds in the following way. (i) One determines $\mathrm{Ric}(K, K)$ and $\hat{\sigma}$ along λ up to order $n - 3$. (ii) With these informations (7) and (8) determine the function D up to order n. (iii) By inversion one gets the parameter s as a function of $D(s)$ up to order n. (iv) Inserting the result into the n^{th} order Taylor expansion of (6) gives the desired z-D-relation up to n^{th} order.

In the following we want to carry through this procedure for $n = 3$. This is considerably simpler than the determination of the higher order terms since, for $n = 3$, we need the shear $\hat{\sigma}$ in zeroth order only, i.e., we only have to know that, as mentioned above, the shear has a zero at $s = 0$. With this information we get the following representation for $D(s)$ from (7) and (8).

$$D(s) = s - \tfrac{1}{12}\,\mathrm{Ric}\big(K(0), K(0)\big)\, s^3 + O\big(s^4\big)\,. \tag{10}$$

Thus, the Taylor expansion of the inverse function $D(s) \longmapsto s$ reads

$$s = D(s) + \tfrac{1}{12}\,\mathrm{Ric}\big(K(0), K(0)\big)\, D(s)^3 + O\big(D(s)^4\big)\,. \tag{11}$$

On the other hand, the general redshift formula (6) implies that

$$\begin{aligned} z(s) = g\big(\nabla_{K(0)} V, K(0)\big)\, s + \tfrac{1}{2}\, g\big(\nabla_{K(0)} \nabla_K V, K(0)\big)\, s^2 + \\ + \tfrac{1}{6}\, g\big(\nabla_{K(0)} \nabla_K \nabla_K V, K(0)\big)\, s^3 + O\big(s^4\big)\,. \end{aligned} \tag{12}$$

Inserting (11) into (12) gives us the following representation for the first three coefficients in the z-D-relation (9).

$$H_1 = g\big(\nabla_{K(0)} V, K(0)\big)\,, \tag{13}$$

$$H_2 = \tfrac{1}{2}\, g\big(\nabla_{K(0)} \nabla_K V, K(0)\big)\,, \tag{14}$$

$$H_3 = \tfrac{1}{6}\, g\big(\nabla_{K(0)} \nabla_K \nabla_K V, K(0)\big) \;+\; \tfrac{1}{12}\, g\big(\nabla_{K(0)} V, K(0)\big)\, \mathrm{Ric}\big(K(0), K(0)\big)\,. \tag{15}$$

In Sections 3, 4 and 5 we shall use these representations to characterize first, second and third order Hubble models. Luckily enough it will turn out that knowledge of the first three coefficients H_1, H_2 and H_3 is already sufficient to characterize Hubble models of arbitrary order.

Series expansions of the kind used here are often applied in cosmology. Contrary to other authors, in particular to observational astronomers, we deliberately decided to express z as a function of D rather than the other way round. The reason is that the function $s \longmapsto D(s)$ is always invertible near $s = 0$ whereas this is not true, in general, for the function $s \longmapsto z(s)$.

2.3 Luminosity distance and reciprocity law

As an alternative to the angular diameter distance, one could also use the so-called *luminosity distance* which is based on the intuitive idea that the farther away an object is the fainter it appears. Contrary to the angular diameter distance, the luminosity distance is not a purely geometric quantity. For that reason, one often uses a so-called *corrected luminosity distance*. The latter is defined, for each $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$, as a function $\tilde{D} : [0, s_0] \longrightarrow \mathbb{R}$ in the following way. We fix a parameter value $s \in [0, s_0]$ and consider an infinitesimally thin bundle of light rays around λ with vertex at $\lambda(s)$. We can then consider the ratio of the cross-sectional area of this bundle at $\lambda(0)$ and its opening angle (solid angle) at $\lambda(s)$, measured in the rest space of the respective V-observer. The square-root of this ratio is, by definition, the corrected luminosity distance $\tilde{D}(s)$. Thus, the definition of $\tilde{D}(s)$ is quite analogous to the definition of $D(s)$, with the role of source and observer interchanged. The (uncorrected) luminosity distance $\hat{D}$ differs from the corrected luminosity distance $\tilde{D}$ by a redshift factor $1 + z(s)$,

$$\hat{D}(s) = \big(1 + z(s)\big)\, \tilde{D}(s)\,. \tag{16}$$

The (uncorrected) luminosity distance relates, for an isotropically radiating light source at $\lambda(s)$, the energy flux at $\lambda(0)$ to the total intensity of the light source at $\lambda(s)$. The additional factor $1 + z(s)$ takes care of the fact that the energy (= frequency) of each photon undergoes a redshift. Thus, the (uncorrected) luminosity distance is an energetic quantity whereas the corrected luminosity distance is a purely geometric quantity. For a more detailed discussion of these two notions we refer to Schneider, Ehlers and Falco [24], Section 3.5.

It is a remarkable fact that along every light ray the angular diameter distance $D(s)$, the corrected luminosity distance $\tilde{D}(s)$, and the redshift $z(s)$ are related by the universal *reciprocity law*

$$\tilde{D}(s) = \big(1 + z(s)\big) D(s)\,, \tag{17}$$

which can be rewritten, with the help of (16), in terms of the (uncorrected) luminosity distance. This law was discovered in 1933 by Etherington [3], a proof can be found, e.g., in Ellis [8] or in Schneider, Ehlers and Falco [24]. The name “reciprocity law” dates back to a 1902 paper by Straubel [1] who proved an analogous law for light rays in isotropic media (in ordinary optics, of course).

The reciprocity law implies that D and $\tilde{D}$ coincide along every ray if and only if the model is redshift-free. The latter case was characterized by Theorem 2.3.

In arbitrary kinematical world models, (17) can be used to replace $D(s)$ with $\tilde{D}(s)$ whenever this is desired. In particular, we can use (17) to rewrite the z-D-relation as a z-$\tilde{D}$-relation. Thereby it is readily verified that the first n coefficients $H_1, \dots, H_n$ of the z-D-relation also determine the z-$\tilde{D}$-relation up to order n. In the following we work out the precise relation for the case $n = 3$ for which the coefficients are given by (13), (14) and (15). To that end we first replace $z(s)$ in (17) by the representation (9) . This gives

$$\tilde{D}(s) = D(s) + H_1\, D(s)^2 + H_2\, D(s)^3 + O\big(D(s)^4\big)\,, \tag{18}$$

and, upon solving for $D(s)$,

$$D(s) = \tilde{D}(s) - H_1\, \tilde{D}(s)^2 + \big(2\, H_1^2 - H_2\big)\, \tilde{D}(s)^3 + O\big(\tilde{D}(s)^4\big)\,. \tag{19}$$

This expression for $D(s)$ can now be reintroduced into (9),

$$z(s) = H_1\, \tilde{D}(s) + \big(H_2 - H_1^2\big)\, \tilde{D}(s)^2 + \big(2\, H_1^3 - 3\, H_1\, H_2 + H_3\big)\, \tilde{D}(s)^3 + O\big(\tilde{D}(s)^4\big)\,, \tag{20}$$

thereby yielding the desired z-$\tilde{D}$-relation up to third order. − An analogous calculation, using (16), allows to rewrite this as a z-$\hat{D}$-relation in the following form.

$$z(s) = H_1\, \hat{D}(s) + \big(H_2 - 2\, H_1^2\big)\, \hat{D}(s)^2 + \big(7\, H_1^3 - 6\, H_1\, H_2 + H_3\big)\, \hat{D}(s)^3 + O\big(\hat{D}(s)^4\big)\,. \tag{21}$$

2.4 Example: Robertson-Walker models

A kinematical world model $(\mathcal{M}, g, V)$ is a *Robertson-Walker model* if (i) $\mathcal{M}$ can be foliated into hypersurfaces perpendicular to V which are Riemannian manifolds of constant curvature; (ii) V is proportional to a conformal Killing vector field whose Lorentz length is constant on each hypersurface perpendicular to V.

From the symmetry of a Robertson-Walker model it follows immediately that the functions z, D and $\tilde{D}$ take the same form for two light rays λ_1 and λ_2 in $\mathcal{L}(\mathcal{M}, g, V)$ whenever the initial points $\lambda_1(0)$ and $\lambda_2(0)$ lie on a common hypersurface perpendicular to V. This implies that the z-D-relation of a Robertson-Walker model is isotropic over the whole parameter interval on which the light rays are defined. In particular, Robertson-Walker models are n^{th} order Hubble models for all $n \in \mathbb{N}$.

For Friedmann-Robertson-Walker models, i.e., for Robertson-Walker models that satisfy Einstein's field equation for dust, the z-D-relation can be found in many text-books on cosmology, see, e.g., Raychaudhuri [13], p. 61. For arbitrary Robertson-Walker models, however, it is not possible to write the z-D-relation in closed form.

In the following we derive the first three coefficients H_1, H_2 and H_3 for an arbitrary Robertson-Walker model. To that end we introduce coordinates $(\chi, \varphi, \vartheta, t)$ with $V = \partial_t$ such that the Robertson-Walker metric takes the familiar form

$$g = S(t)^2 \Big(d\chi \otimes d\chi + \Sigma(\chi)^2 \big(d\vartheta \otimes d\vartheta + \sin^2\!\vartheta\, d\varphi \otimes d\varphi \big) \Big) \; - \; dt \otimes dt\,. \tag{22}$$

Here S is a strictly positive but otherwise unspecified function of t and $\Sigma(\chi)$ is equal to one of the three functions $\sin\chi$, χ, $\sinh\chi$, depending on whether the hypersurfaces perpendicular to V have positive, zero, or negative curvature.

In this representation the conformal Killing vector field proportional to V takes the form $W = S(t)\, V$. Thus, Theorem 2.2 immediately gives us the following expression for the redshift $z(s)$.

$$z(s) = \frac{S(t)_{\lambda(0)}}{S(t)_{\lambda(s)}} \; - \; 1\,. \tag{23}$$

The angular diameter distance $D(s)$ cannot be determined in closed form for arbitrary Robertson-Walker models. To get a series expression, we first observe that the Robertson-Walker spacetime $(\mathcal{M}, g)$ is conformally flat. As mentioned above, this implies that the shear $\hat{\sigma}$ vanishes for each light ray $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$, so (7) takes the form

$$D''(s) = -\tfrac{1}{2}\, \mathrm{Ric}\big(K(s), K(s)\big)\, D(s)\,. \tag{24}$$

Owing to the symmetry of the Robertson-Walker spacetime it suffices to consider light rays $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ such that the point $\lambda(0)$ is at the origin of the coordinate system (i.e., at $\chi = 0$) and the coordinates ϑ and φ are constant along λ. Then the tangent field $K = \lambda'$ takes the form

$$K(s) = S(t)_{\lambda(0)} \Big(\frac{1}{S(t)^2}\, \partial_\chi - \frac{1}{S(t)}\, \partial_t \Big)_{\lambda(s)} . \tag{25}$$

With the Ricci tensor of the metric (22), which can be found in any text-book on general relativity, this yields

$$\mathrm{Ric}\big(K(s), K(s)\big) \,=\, 2\, S(t)^2_{\lambda(0)} \Big(- \frac{\ddot{S}(t)}{S(t)^3} + \frac{\dot{S}(t)^2}{S(t)^4} + \frac{\varepsilon}{S(t)^4} \Big)_{\lambda(s)} , \tag{26}$$

where ε takes the value $1, 0, -1$, depending on whether the hypersurfaces perpendicular to V have positive, zero or negative curvature. Please note that the overdot in (26) denotes derivative with respect to the coordinate function t. If we want to calculate derivatives of (26) with respect to the curve parameter s we have to use the formula

$$\frac{d}{ds}\, t\big(\lambda(s)\big) = dt\big(K(s)\big) = -g\big(V_{\lambda(s)}, K(s)\big) = -\frac{S(t)_{\lambda(0)}}{S(t)_{\lambda(s)}} \tag{27}$$

where the first equality follows from the definition of K, the second from (22) and the third from (25). With the help of (27) we can determine from (24) and (8) as many terms in the Taylor expansion of $D(s)$ as we wish. Upon solving for s, the result can then be inserted into (23) to get the z-D-relation in Robertson-Walker models to arbitrarily high order. Here we only give the first three coefficients which, quite generally, are determined by (13), (14) and (15).

$$H_1 = \Big(\frac{\dot{S}(t)}{S(t)} \Big)_{\lambda(0)} , \tag{28}$$

$$H_2 = \Big(- \frac{\ddot{S}(t)}{2\, S(t)} + \frac{3\, \dot{S}(t)^2}{2\, S(t)^2} \Big)_{\lambda(0)} , \tag{29}$$

$$H_3 = \Big(\frac{\dddot{S}(t)}{6\, S(t)} - \frac{11\, \ddot{S}(t)\, \dot{S}(t)}{6\, S(t)^2} + \frac{8\, \dot{S}(t)^3}{3\, S(t)^3} + \frac{\varepsilon\, \dot{S}(t)}{6\, S(t)^3} \Big)_{\lambda(0)} . \tag{30}$$

(28), (29) and (30) give us the z-D-relation, the z-$\tilde{D}$-relation and the z-$\hat{D}$-relation in Robertson-Walker models up to third order, via (9), (20) and (21).

The coefficient H_2 is usually expressed in terms of the "Hubble constant" H_1 and the "deceleration parameter"

$$q = -\Big(\frac{\ddot{S}(t) S(t)}{\dot{S}(t)^2} \Big)_{\lambda(0)} , \tag{31}$$

according to

$$H_2 = \frac{q+3}{2} H_1^2 . \tag{32}$$

The name "deceleration parameter" refers to the fact that, for an expanding Robertson-Walker model, a positive value of q indicates that the expansion is decelerated whereas a negative value of q indicates that it is accelerated. Please note that (32) can be used to define q in arbitrary kinematical world models for all light rays for which $H_1 \neq 0$.

3 First order Hubble models

In this section we want to characterize first order Hubble models. (Please recall Definition 2.4.) To that end we have to consider the coefficient H_1 which, according to (13), is determined by the covariant derivative of V. In the following we assume that the reader is familiar with the decomposition of this covariant derivative into rotation ω, shear σ, expansion θ and acceleration $\nabla_V V$ (see, e.g., Hawking and Ellis [10], Section 4.1),

$$\begin{aligned} g\big(\nabla_X V, Y\big) &= \omega\big(X, Y\big) + \sigma\big(X, Y\big) + \\ &+ \tfrac{1}{3}\, \theta \Big(g\big(X, Y\big) + g\big(X, V\big)\, g\big(Y, V\big) \Big) - g\big(X, V\big)\, g\big(\nabla_V V, Y\big) , \end{aligned} \tag{33}$$

where X and Y denote arbitrary vector fields on $\mathcal{M}$. For the tangent field $K = \dot{\lambda}$ of a light ray $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ this yields

$$g\big(\nabla_K V, K\big) = \sigma\big(K, K\big) + \tfrac{1}{3}\,\theta\, g\big(K, V\big)^2 - g\big(K, V\big)\, g\big(\nabla_V V, K\big)\,. \tag{34}$$

With the initial condition (3) , (34) can be evaluated at the initial point $s = 0$. Inserting this into (13) yields the following representation of the coefficient H_1.

$$H_1 = \sigma_p\big(K(0), K(0)\big) + \tfrac{1}{3}\,\theta(p) - g\big(\nabla_{V_p} V, K(0)\big)\,. \tag{35}$$

From this equation we read that a non-vanishing shear would produce a quadrupole anisotropy whereas a non-vanishing acceleration would produce a dipole anisotropy. This proves the following result which is known since Ehlers [5].

Theorem 3.1. *A kinematical world model $(\mathcal{M}, g, V)$ admits an isotropic Hubble law of first order at p if and only if the shear and the acceleration of V vanish at p. At each point where this is true the "Hubble constant" is then given by $H_1 = \frac{\theta}{3}$ where θ denotes the expansion of V.*

Hence, $(\mathcal{M}, g, V)$ is a first order Hubble model if and only if V is everywhere shear-free and freely falling. This is true, e.g., in Robertson-Walker models, but also in many other spacetimes. In particular, the redshift-free models, which are characterized by Theorem 2.3, are (somewhat trivial) examples of first order Hubble models. In Section 4 below we shall illustrate by an example that a first order Hubble model (and even a second order Hubble model) may be expanding and rotating.

For explicit calculations it is often recommendable to use local coordinates (x^1, x^2, x^3, x^4) on $\mathcal{M}$ such that the distinguished observer field takes the form $V = \partial_4$. Owing to the normalization condition $g(V, V) = -1$, this puts the metric into the form

$$g = g_{ik}\, dx^i \otimes dx^k + g_{j4}\big(dx^j \otimes dx^4 + dx^4 \otimes dx^j\big) - dx^4 \otimes dx^4\,. \tag{36}$$

Here and in the following we use Einstein's summation convention with latin indices running from 1 to 3. It is an easy exercise to verify that, in such a coordinate system, the rotation ω, the shear σ, the expansion θ and the acceleration $\nabla_V V$ of V are given by the following equations.

$$\omega = \tfrac{1}{2}(\partial_i g_{j4} - \partial_j g_{i4})\, dx^j \otimes dx^i\,, \tag{37}$$

$$\sigma = \tfrac{1}{2}\big(\partial_4 g_{ij} - \tfrac{2}{3}\theta(g_{ij} + g_{i4}\, g_{j4}) + g_{i4}\, \partial_4 g_{j4} + g_{j4}\, \partial_4 g_{i4}\big)\, dx^i \otimes dx^j\,, \tag{38}$$

$$\theta = \tfrac{1}{2}\, g^{ij}\, \partial_4 g_{ij}\,, \tag{39}$$

$$g\big(\nabla_V V,\, \cdot\,\big) = \partial_4 g_{i4}\, dx^i\,. \tag{40}$$

Hence, the condition that both the shear and the acceleration vanish is equivalent to

$$\partial_4 g_{i4} = 0\,, \tag{41}$$

$$\partial_4 g_{ij} = \tfrac{2}{3}\,\theta\,(g_{ij} + g_{i4}\, g_{j4})\,. \tag{42}$$

Please note that (39) is a consequence of (42). Hence, whenever (42) is satisfied with some function θ, this function necessarily gives the expansion of V. We have thus proven the following result.

Theorem 3.2. *$(\mathcal{M}, g, V)$ is a first order Hubble model if and only if the following holds true. Near each point there is a coordinate system with $V = \partial_4$, putting the metric into the form of equation* (36)*, such that the metric coefficients satisfy the differential equations* (41) *and* (42) *with some function θ. This function θ is then equal to the expansion of V.*

This theorem gives the following construction method for first order Hubble models. We may freely choose a function $\theta(x^1, x^2, x^3, x^4)$ and we may freely choose initial values $g_{i4}(x^1, x^2, x^3, 0)$ and $g_{ij}(x^1, x^2, x^3, 0)$. Then the differential equations (41) and (42) determine a unique solution g_{i4}, g_{ij} on some neighborhood of the initial hypersurface $x^4 = 0$. Inserting this solution into (36) yields a metric of Lorentzian signature on some (possibly smaller) neighborhood of the initial hypersurface $x^4 = 0$ provided that the initial values have been chosen in accordance with this signature. By construction, the resulting model is shear-free and acceleration-free and its expansion is given by the prescribed function θ. From (37) and (41) we read that the rotation is completely determined by the initial values for g_{i4}. Thus, it is easy to construct models with non-vanishing rotation.

Every first order Hubble model can be constructed in this way, at least locally. In the next section we shall illustrate this method with an example. – For later purpose we establish the following result.

Theorem 3.3. *Let $(\mathcal{M}, g, V)$ be a first order Hubble model. Then the rotation ω of V satisfies the equations $\omega = -d\big(g(V,\cdot)\big)$ and $L_V \omega = 0$. Here L_V denotes the Lie derivative in the direction of V.*

Proof: These two equations follow from $\nabla_V V = 0$ which, by Theorem 3.1, holds for first order Hubble models. This is easily verified in a chart with $V = \partial_4$, using (37) and (40). ∎

The equation $L_V \omega = 0$ may be interpreted as saying that a non-rotating first order Hubble model cannot become rotating in the course of time, and vice versa.

In the rest of this section we investigate under what condition a first order Hubble model admits a redshift potential. (Please recall Theorem 2.2.) Related results can be found in Section VII of Perlick [23] where a slightly different terminology is used.

Theorem 3.4. *Let $(\mathcal{M}, g, V)$ be a first order Hubble model. Then a function $f : \mathcal{M} \longrightarrow \mathbb{R}$ is a redshift potential for $(\mathcal{M}, g, V)$ if and only if its differential satisfies the equation $3\, df = -\,\theta\, g(V,\,\cdot\,)$. For the existence of such a function the conditions $d\theta \wedge g(V,\,\cdot\,) = 0$ and $\theta\, \omega = 0$ are necessary and, on simply connected regions, also sufficient. (Here $\wedge$ denotes the antisymmetrized tensor product.)*

Proof: By Theorem 3.2, the differential equations (41) and (42) have to be satisfied in coordinates of the kind indicated there. In coordinate-free notation, this condition can be rewritten as

$$\big(L_V g\big)\big(X,Y\big) = \tfrac{2}{3}\,\theta\,\Big(g\big(X,Y\big) + g\big(X,V\big)\,g\big(Y,V\big)\Big)\,, \tag{43}$$

where L_V is the Lie derivative in the direction of V and X and Y are arbitrary vector fields. Using standard derivation rules, we find from (43) that for every function f the equation

$$\begin{aligned} e^{-f}\,\big(L_{e^f V} g\big)\big(X,Y\big) = \tfrac{2}{3}\,\theta\,\Big(g\big(X,Y\big) + g\big(X,V\big)\,g\big(Y,V\big)\Big) + \\ + g\big(X,V\big)\,df\big(Y\big) + g\big(Y,V\big)\,df\big(X\big) \end{aligned} \tag{44}$$

has to hold. By Theorem 2.2, f is a redshift potential if and only if the left-hand side of (44) is of the form $h\,g(X,Y)$ with some function h. By decomposing X and Y into components parallel to V and perpendicular to V we find that this is true if and only if $3\,df = -\,\theta\,g(V,\,\cdot\,)$ and that h is then necessarily equal to $2\,df(V)$. – For the existence of a function f with $3\,df = -\,\theta\,g(V,\,\cdot\,)$ the condition $d\big(\theta\,g(V,\,\cdot\,)\big) = 0$ is necessary. On simply connected regions it is also sufficient, owing to the well-known Poincaré lemma. Using the product rule for the exterior derivative, this condition takes the form $d\theta \wedge g(V,\,\cdot\,) + \theta\, d\big(g(V,\,\cdot\,)\big) = 0$. Since $(\mathcal{M}, g, V)$ is a first order Hubble model and, thus, acceleration-free, we can read from the coordinate expressions (37) and (40) that $-\,d\big(g(V,\,\cdot\,)\big)$ is equal to the rotation ω, i.e., the condition for the existence of a redshift potential takes the form $d\theta \wedge g(V,\,\cdot\,) - \theta\,\omega = 0$. Applying this to vectors orthogonal to V shows that the two terms have to vanish seperately. ∎

Hence, a first order Hubble model with non-vanishing expansion and non-vanishing rotation cannot admit a redshift potential, i.e., it cannot be conformally stationary. Clearly, the equation $3\,df = -\,\theta\,g(V,\,\cdot\,)$ holds with $f = \text{const.}$ if and only if $\theta = 0$. This proves the following corollary.

Theorem 3.5. *A first order Hubble model $(\mathcal{M}, g, V)$ is redshift-free if and only if the expansion of V vanishes.*

4 Second order Hubble models

According to Definition 2.4, a second order Hubble model is a first order Hubble model for which, in addition, the coefficient H_2 is isotropic. To find a characterization of second order Hubble models we first derive a formula for the coefficient H_2 that holds in first order Hubble models and then we investigate under what conditions this coefficient is isotropic.

In the preceding section we have seen that first order Hubble models are shear-free and freely falling. This implies that the general formula (34) for the tangent field $K = \lambda'$ of a light ray $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ simplifies to

$$g\big(\nabla_K V, K\big) = \tfrac{1}{3}\,\theta\,g\big(K,V\big)^2\,. \tag{45}$$

By differentiation with respect to the curve parameter we get

$$g\big(\nabla_K \nabla_K V, K\big) = \tfrac{1}{3}\,d\theta(K)\,g\big(K,V\big)^2 + \tfrac{2}{9}\,\theta^2\,g\big(K,V\big)^3\,. \tag{46}$$

Evaluating at $s = 0$, with the initial condition (3), and inserting the result into (14) yields the following representation of the coefficient H_2 for first order Hubble models.

$$H_2 = \tfrac{1}{6}\, d\theta\big(K(0)\big) + \tfrac{1}{9}\,\theta(p)^2 \,. \tag{47}$$

From this equation we read that there is a dipole anisotropy unless $d\theta$ vanishes on all vectors perpendicular to V. We have thus proven the following result.

Theorem 4.1. *$(\mathcal{M}, g, V)$ is a second order Hubble model if and only if V is shear-free, freely falling, and the expansion θ satisfies the condition $d\theta \wedge g(V, \cdot\,) = 0$, where $\wedge$ denotes the antisymmetrized tensor product. The coefficients H_1 and H_2 are then given at each point by $H_1 = \frac{1}{3}\,\theta$ and $H_2 = \frac{1}{6}\, d\theta(V) + \frac{1}{9}\,\theta^2$.*

Without further calculation, comparison of this result with Theorem 3.4 yields the following two corollaries.

Theorem 4.2. *A first order Hubble model that admits a redshift potential is a second order Hubble model.*

Theorem 4.3. *Let $(\mathcal{M}, g, V)$ be a second order Hubble model. Then for the existence of a redshift potential the condition $\theta\,\omega = 0$ is necessary and, on simply connected regions, also sufficient.*

Another consequence of Theorem 4.1 is the following fact that will be needed later on.

Theorem 4.4. *In a second order Hubble model, the differential $d\big(d\theta(V)\big)$ vanishes on all vectors perpendicular to V.*

Proof: By Theorem 4.1, the condition $d\theta \wedge g(V, \cdot) = 0$ is satisfied. This is equivalent to $d\theta = -d\theta(V)\, g(V, \cdot)$. Application of the exterior derivative, together with Theorem 3.3, yields $0 = -d\big(d\theta(V)\big) \wedge g(V, \cdot) + d\theta(V)\,\omega$. Now we only have to insert V into the first slot and a vector perpendicular to V into the second slot of this two-form equation to get the desired result. ■

We now turn to the interesting question of whether a second order Hubble model can be both rotating and expanding. To that end we prove the following theorem.

Theorem 4.5. *Let $(\mathcal{M}, g, V)$ be a second order Hubble model. If the differential $d\theta$ of the expansion of V is non-zero at a point p and, thus, on some neighborhood $\mathcal{U}$ of p, then V is irrotational on $\mathcal{U}$.*

Proof: If $d\theta \neq 0$ at p, the function θ foliates a neighborhood $\mathcal{U}$ of p into hypersurfaces. The condition $d\theta \wedge g(V, \cdot\,) = 0$ then guarantees that V is perpendicular to those hypersurfaces $\theta = \text{const.}$, i.e., that V is irrotational on $\mathcal{U}$. ■

In other words, a second order Hubble model can be rotating only if the expansion θ is a constant. In the preceding section we have used Theorem 3.2 to establish a construction method for first order Hubble models in which the expansion θ could be prescribed arbitrarily. This method can be used, with $\theta = \text{const.}$, to construct rotating and expanding second order Hubble models. Here is an example.

Choose $\theta(x^1, x^2, x^3, x^4) = 3/(2t_o)$ with some non-zero constant t_o and initial conditions of the form $g_{i4}(x^1, x^2, x^3, 0) = F(x^2)\,\delta_i^1$ and $g_{ij}(x^1, x^2, x^3, 0) = \delta_{ij}$. Using the notation $(x^1, x^2, x^3, x^4) = (x, y, z, t)$, the resulting metric reads

$$\begin{aligned} g = e^{t/t_o}\left(dx \otimes dx + dy \otimes dy + dz \otimes dz\right) + F(y)^2 \left(e^{t/t_o} - 1\right) dx \otimes dx + \\ + F(y)\big(dx \otimes dt + dt \otimes dx\big) - dt \otimes dt \,. \end{aligned} \tag{48}$$

This is a second order Hubble model with non-vanishing expansion and non-vanishing rotation. The expansion takes the prescribed value $\theta = 3/(2t_o)$ and the rotation, according to (37), is given by $\omega = F'(y)\, dx \wedge dy$. The latter is, indeed, non-zero unless F is constant. For $F = 0$ (48) is the well-known *de Sitter metric*. (More precisely, our coordinate system covers one half of de Sitter spacetime which is known as the *steady state universe*.) In this sense, our example can be viewed as a rotating generalization of the de Sitter metric.

Near points with $d\theta \neq 0$, second order Hubble models are necessarily irrotational. In terms of local coordinates, irrotational second order Hubble models are characterized by the following theorem.

Theorem 4.6. *Let $(\mathcal{M}, g, V)$ be a second order Hubble model and assume that the rotation ω of V vanishes on some neighborhood of a point $p \in \mathcal{M}$. Then there is a local coordinate system (x^1, x^2, x^3, x^4) near p such that $V = \partial_4$ and $g = g_{ij}\, dx^i \otimes dx^j - dx^4 \otimes dx^4$ in which the metric coefficients g_{ij} satisfy the differential equation $\partial_4 g_{ij} = \frac{2}{3}\,\theta\, g_{ij}$. In this coordinate system the expansion θ is independent of the spatial coordinates, $\partial_i\theta = 0$.*

Proof: On some neighborhood of p we choose a coordinate system with $V = \partial_4$. From the preceding section we know that in such a coordinate system, quite generally, the rotation ω of V is given by (37). Since, by assumption, ω vanishes near p, the well-known Poincaré lemma guarantees that on a (sufficiently small, simply connected) neighborhood of p there must be a function h such that $\partial_i h = g_{i4}$. But then a coordinate transformation of the form $(x^1, x^2, x^3, x^4) \longmapsto (x^1, x^2, x^3, x^4 - h)$ transforms g_{i4} to zero. In the new coordinates, the metric (36) and the 4-derivative of its spatial components (42) take the desired form. The equation $\partial_i \theta = 0$ follows from $d\theta \wedge g(V, \cdot\,) = 0$, please recall Theorem 4.1. ■

Thus, if it is our goal to (locally) determine all non-rotating second order Hubble models, our construction method takes the following simplified form. We have to prescribe a function θ with $\partial_i \theta = 0$ and we have to prescribe initial values $g_{ij}(x^1, x^2, x^3, 0)$. Then we have to solve the differential equation $\partial_4 g_{ij} = \frac{2}{3}\,\theta\, g_{ij}$ with these initial values, thereby getting the desired metric in the form $g = g_{ij}\, dx^i \otimes dx^j - dx^4 \otimes dx^4$.

5 Third and higher order Hubble models

As indicated already in the introduction, the investigation of third order Hubble models will lead our discussion to a natural end. We shall see that fourth and higher order isotropy would give no further restrictions, i.e., that third order Hubble models have already maximal isotropy properties as far as the z-D-relation is concerned.

To work this out, we have to derive a formula for H_3 that holds in second order Hubble models and we have to investigate under which additional conditions this coefficient is isotropic. From Theorem 4.1 we know that in a second order Hubble model the differential of the expansion $d\theta$ must be a multiple of $g(V, \cdot\,)$. This puts the formula (46) for the tangent field $K = \lambda'$ of an arbitrary light ray $\lambda \in \mathcal{L}(\mathcal{M}, g, V)$ into the form

$$g\big(\nabla_K \nabla_K V, K\big) = -\,\tfrac{1}{3}\, d\theta(V)\, g\big(K, V\big)^3 + \tfrac{2}{9}\,\theta^2\, g\big(K, V\big)^3 \,. \qquad (49)$$

We now calculate the derivative of this equation with respect to the curve parameter s at the point $s = 0$ and we use the initial condition (3). Inserting the result into (15) and using Theorem 4.4 yields the following representation of the coefficient H_3 for second order Hubble models.

$$H_3 = \tfrac{1}{18} d\big(d\theta(V)\big)\big(V\big) - \tfrac{7}{54}\,\theta\, d\theta(V) + \tfrac{1}{27}\,\theta^3 + \tfrac{1}{36}\,\theta\, \mathrm{Ric}\big(K(0), K(0)\big) \,. \qquad (50)$$

From this equation we read that there is a dipole anisotropy unless the one-form $\theta\, \mathrm{Ric}(V, \cdot\,)$ vanishes on all vectors perpendicular to V and that there is a quadrupole anisotropy unless on the orthocomplement of V the second rank tensor $\theta\, \mathrm{Ric}$ is a multiple of the metric. In combination with Theorem 4.1 this proves the following theorem.

Theorem 5.1. *A kinematical world model $(\mathcal{M}, g, V)$ is a third order Hubble model if and only if the following conditions are satisfied.* (i) *V is shear-free and freely falling.* (ii) *The one-forms $d\theta$ and $\theta\, \mathrm{Ric}(V, \cdot\,)$ vanish on all vectors perpendicular to V.* (iii) *There is a function k on $\mathcal{M}$ such that $\theta\, \mathrm{Ric}(X, Y) = k\, g(X, Y)$ for all vector fields X and Y perpendicular to V.*

We shall now prove that these three conditions (i), (ii) and (iii) are satisfied only in redshift-free models and in Robertson-Walker models (and in models composed thereof). The following theorem is at the focus of our investigation.

Theorem 5.2. *In a third order Hubble model $(\mathcal{M}, g, V)$, every point $p \in \mathcal{M}$ has a neighborhood $\mathcal{U}$ such that $(\mathcal{U}, g|_{\mathcal{U}}, V|_{\mathcal{U}})$ is redshift-free or a Robertson-Walker model.*

Proof: We denote by $\mathcal{V}$ the set of all points p in $\mathcal{M}$ such that the rotation ω of V is non-zero at p. Moreover, we denote by $\overline{\mathcal{V}}$ the closure of $\mathcal{V}$ in $\mathcal{M}$ and by $\mathcal{W}$ the union of all open sets that are contained in $\overline{\mathcal{V}}$. Clearly, $\mathcal{V}$ and $\mathcal{W}$ are (possibly empty) open subsets of $\mathcal{M}$ with $\mathcal{V} \subseteq \mathcal{W}$ and $\overline{\mathcal{V}} = \overline{\mathcal{W}}$. To prove that any point $p \in \mathcal{M}$ admits a neighborhood with one of the desired properties we distinguish the following three cases. (a) $p \in \mathcal{W}$. (b) $p \in \mathcal{M} \setminus \overline{\mathcal{W}}$. (c) $p \notin \mathcal{W}$ and $p \notin \mathcal{M} \setminus \overline{\mathcal{W}}$. – In case (a) we can find a neighborhood $\mathcal{U}$ of p such that ω is different from zero almost everywhere on $\mathcal{U}$. If we choose $\mathcal{U}$ connected, Theorem 4.5 implies that θ must be a constant on $\mathcal{U}$. If this constant is zero we are done since then, by Theorem 3.5, the model is redshift-free on $\mathcal{U}$. We shall now assume that this constant is non-zero and we shall show that this assumption leads to a contradiction. If θ is a non-zero constant, we can read from conditions (ii) and (iii) of Theorem 5.1 that $(\mathcal{M}, g)$ is a perfect fluid solution of Einstein's field equation with V giving the four-velocity of the fluid. By property (i) of Theorem 5.1, we know that V is shear-free and geodesic. For a perfect fluid solution with these properties the product of expansion and rotation has to vanish, as is proven as Theorem 1 in Senovilla, Sopuerta and Szekeres [29]. Since by assumption θ is a non-zero constant and ω is non-zero almost everywhere on $\mathcal{U}$, this gives the desired contradiction. – We now turn to case (b). If θ is identically zero on some neighborhood of p, we are done since then Theorem 3.5 implies that the model is redshift-free near p. So we assume henceforth that there is no neighborhood of p on which θ is identically zero. We want to show that in this case the model must be a Robertson-Walker model near p. Our assumption $p \in \mathcal{M} \setminus \overline{\mathcal{W}}$ is tantamount to the condition of ω vanishing identically on some neighborhood $\mathcal{U}$ of p. By Theorem 4.3, this condition guarantees the existence of a redshift potential f on some (sufficiently small, simply connected) neighborhood $\mathcal{U}$ of p. The same condition guarantees that $\mathcal{U}$ can be foliated into hypersurfaces orthogonal to V. Since, by Theorem 3.4, the redshift potential f satisfies the equation $3\,df = -\,\theta\, g(V, \cdot)$, it must be constant on each of those hypersurfaces orthogonal to V. In combination with Theorem 2.2 this implies that $e^f V$ is a conformal Killing vector field which has constant Lorentz length on each hypersurface orthogonal to V. What remains to be shown is that those hypersurfaces are maximally symmetric, i.e., Riemannian manifolds of constant curvature. It is well known that this is true if and only if $\mathrm{Ric}^{(3)}$ is a scalar multiple of $g^{(3)}$, where $g^{(3)}$ denotes the Riemannian metric induced on the hypersurface and $\mathrm{Ric}^{(3)}$ denotes the Ricci tensor of $g^{(3)}$. $\mathrm{Ric}^{(3)}$ is related to the 4-dimensional Ricci tensor Ric by the well-known *Gauss-Codazzi equations*, see, e.g. Stephani [14], p. 155. In the case at hand, it is most convenient to use the special coordinate representation of Theorem 4.6 for the metric. By writing the Gauss-Codazzi equations in this particular coordinates it is readily shown that, on vectors perpendicular to V, Ric differs from $\mathrm{Ric}^{(3)}$ only by a multiple of the metric. But then condition (iii) of Theorem 5.1 guarantees that $\theta\,\mathrm{Ric}^{(3)}$ is a multiple of $g^{(3)}$. This shows that every hypersurface perpendicular to V is of constant curvature provided that θ is non-zero on this hypersurface. (Please recall that, by property (ii) of Theorem 5.1, θ must be constant on such a hypersurface.) Now we use our assumption that there is no neighborhood of p on which θ is identically zero. Thus, in each neighborhood of p there is a hypersurface perpendicular to V that is of constant curvature. But then the differential equation $\partial_4 g_{ij} = \frac{2}{3}\,\theta\, g_{ij}$ of Theorem 4.6 implies that, on some neighborhood of p, all hypersurfaces perpendicular to V are of constant curvature. – Finally, we consider case (c), i.e., we assume that p lies on the boundary between the open regions $\mathcal{W}$ and $\mathcal{M} \setminus \overline{\mathcal{W}}$. By Theorem 3.3, each of those open regions is invariant under the flow of V, i.e., the boundary between them is ruled by integral curves of V. From the first step of this proof we know that θ vanishes on $\mathcal{W}$. From the second step we know that $\mathcal{M} \setminus \overline{\mathcal{W}}$ is (locally) foliated into hypersurfaces perpendicular to V. By property (ii) of Theorem 5.1, θ is constant on each of those hypersurfaces. By continuity, θ must be zero on each of those hypersurfaces that extends to the boundary of $\mathcal{W}$, i.e., θ must be zero on a whole neighborhood of each point p that lies on that boundary. Owing to Theorem 3.5, this completes the proof. ■

Clearly, Theorem 5.2 implies that for every point p in a third order Hubble model the z-D-relation is isotropic over some finite interval on which the light rays are defined, not just to within a Taylor approximation of third order. Thus, Theorem 5.2 implies, in particular, that every third order Hubble model is an n^{th} order Hubble model for all $n \in \mathbb{N}$.

Please note that it is possible to construct a third order Hubble model which is neither globally redshift-free nor globally a Robertson-Walker model. E.g., one might start with a portion of a Robertson-Walker model with $\theta = 0$ and glue to that a rotating and redshift-free model at the spatial boundary and an expanding Robertson-Walker model at the timelike boundary. Such a construction is, of course, rather contrived and has probably no relevance to physics. Moreover, models which are redshift-free in some open region are usually not considered as realistic cosmological models. In this sense, Robertson-Walker models are the only realistic cosmological models that admit an isotropic Hubble law of third order.

6 Some remarks on the observational situation

According to the results of Sections 3, 4 and 5, the validity of an isotropic Hubble law in a kinematical world model $(\mathcal{M}, g, V)$ is related to the kinematical invariants of the observer field V. In particular, the results of Section 3 imply that an isotropic Hubble law of first order holds only if the shear and the acceleration of V vanish. Correspondingly, observational bounds on the anisotropy of H_1 would imply bounds on the shear and on the acceleration. This argument is well known; it was used already in the pioneering paper of Kristian and Sachs [7] to roughly estimate the cosmic shear on the basis of the observational H_1 data available at that time.

The results of Section 4 and 5 show that a lot of additional information could be gained by taking the next two coefficients H_2 and H_3 into account. In principle, the results of Section 5 give us a new possibility to test how far away our real universe is from a Robertson-Walker model. To that end it would be necessary to determine observational bounds on the anisotropy of the coefficients H_1, H_2 and H_3 for light rays issuing from our position in the universe into the past. Under the reasonable assumption that our position in the universe is not distinguished, this would allow to estimate bounds on the anisotropy of H_1, H_2 and H_3 for observers at other positions in the universe (not too far away from us, at least). In this way, we could demonstrate that our universe must be close to a Robertson-Walker model to within certain error bounds (at least in a neighborhood of our position in the universe).

Unfortunately, this idea cannot be worked out at the time being because our observational knowledge of H_1, H_2 and H_3 is too insecure. This remark applies in particular to the coefficient H_3 about which we know practically nothing; but even for the linear coefficient H_1, what we know about isotropy is not very satisfactory if we compare it, e.g., with what we know about isotropy of the cosmic background radiation (CBR).

Determination of the z-D-relation is difficult for the following reasons. Whereas the redshift z is a directly observable quantity, the determination of D (or $\tilde{D}$ or $\hat{D}$) requires to know the actual size (or the actual brightness) of the observed galaxy. This must be based on theoretical estimates from which systematic errors are difficult to be eliminated. For a discussion of such systematic errors we refer, e.g., to Narlikar [17], Chapter 12. Even after eliminating all systematic errors, the raw data for z and D cannot be used directly for checking the Hubble law. The reason is that both the observer on the Earth and the observed galaxies must be expected to have a non-vanishing velocity (i.e., a *peculiar motion*) with respect to the mean flow of galaxies which was modeled by the vector field V in our theoretical analysis. Thus, before an isotropic or almost isotropic Hubble law could be expected to hold it is necessary to apply some kind of averaging procedure to eliminate the effect of the peculiar motions of the observed galaxies, and it is necessary to correct for the Doppler effect produced by the peculiar motion of the observer. The latter would give rise to a dipole anisotropy of the redshifts of galaxies, quite analogous to the dipole anisotropy of the CBR caused by our motion with respect to the rest system of the CBR. This dipole anisotropy of the CBR, which was detected in 1977 through balloon experiments, indicates a motion of our Local Group of galaxies of about 600 km/s with respect to the CBR, pointing towards the galactic coordinates $l = 277^o$ and $b = 30^o$. This motion is usually interpreted as a result of the combined gravitational pull of the Virgo Cluster and of the Hydro-Centaurus Supercluster (see, e.g., Sandage and Tammann [28]).

Hence, the first step towards checking the isotropy of the Hubble law must be to determine our motion with respect to the mean flow of galaxies by systematically searching for a dipole anisotropy in galaxy redshifts. The first systematic search of this kind was carried through in the mid-1970s by Rubin et al. [11] who considered an all-sky sample of Sc I galaxies with redshifts between $z = 0.012$ and $z = 0.022$. From the measured redshift z and the measured apparent magnitude m they determined for each galaxy the socalled *Hubble modulus* $\mathrm{HM} = \log_{10}(cz) - 0.2\, m$ where c denotes the velocity of light in km/s. If the observed Sc I galaxies have no systematic motion with respect to the mean flow of galaxies, and if their average absolute luminosity is the same in all directions, then a non-vanishing velocity of the observer with respect to the mean flow of galaxies would lead to a dipole distribution of HM over the sky. Rubin et al. found, indeed, a variation of HM across the sky to which a dipole distribution could be fitted, giving a maximal value of $\mathrm{HM}_{\mathrm{antapex}} = 0.995 \pm 0.012$ and a minimal value of $\mathrm{HM}_{\mathrm{apex}} = 0.912 \pm 0.012$. From this result they concluded that our Local Group would have a velocity of about 450 km/s with respect to the mean flow of galaxies, pointing towards the galactic coordinates $l = 163^o$ and $b = -11^o$. Strangely enough, this direction makes an angle of almost 90^o with the preferred axis of the dipole anisotropy of the CBR which was mentioned above, thereby indicating a large systematic drift of the galaxies with respect to the CBR. Clearly, this implication has little aesthetic appeal. It may, therefore, be viewed as reassuring that later studies led to completely different results. In particular, an investigation by Hart and Davies [15], based on 21 cm H I observations of Sbc spiral galaxies at redshifts between $z = 0.003$ and $z = 0.018$, indicated for our Local Group a velocity of 436 km/s towards $l = 264^0$ and $b = 45^o$. Obviously, this result is in much better agreement with the idea that, on average, the galaxies are at rest with respect to the CBR. The controversy about the results of Rubin et al. is reviewed in Section 12.3 of Narlikar [17], in Collins, Joseph and Robertson [19] and in Rubin [20]. For a more recent update of what we know about the motion of our Local Group and of the resulting implications for the Hubble law we refer to Sandage and Tammann [28].

It should be mentioned that the absolute value of H_1 (averaged over the sky) is of no relevance for our argument. The reader is probably familiar with the long-drawn-out debate whether H_1 is close to 50 (km/s)/Megaparsec or close to 100 (km/s)/Megaparsec. (Please note that in our notation H_1 has, indeed, the dimension of an inverse time if D is measured in light seconds.) Recent data, partly found with the

Hubble Space Telescope, give some hope that this debate is going to be settled fairly soon, at a value close to 65 (km/s)/Megaparsec. In particular, this value is favored by very recent observations of type Ia supernovae in distant galaxies, see Riess et al. [30]. If it is true that, with the help of new observational methods, the value of H_1 (averaged over the sky) can be determined with satisfactory accuracy, then there is some chance that, with the help of the same observational methods, our knowledge about the isotropy of H_1 is going to increase in the future.

As to the second coefficient H_2, the situation is much worse than as to H_1. This should not come as a surprise since only a small portion of galaxies for which the distance can be determined with sufficient reliability are so far away from us that they lie beyond the linear regime of the z-D-relation. To link up with standard notation of astrophysics we have to express our coefficient H_2 in terms of the "Hubble constant" H_1 and the "deceleration parameter" q via equation (32). No reliable estimates of the isotropy of q have been determined so far, and the value of q (averaged over the sky) is still dubious. The first systematic studies to determine this value date back to Kristian, Sandage and Westphal [12]. Later investigations, using VLBI measurements of compact radio sources associated with active galaxies and quasars, seemed to indicate a value of q close to 0.5, see, e.g., Kellermann [25]. For Friedmann-Robertson-Walker models, $q = 0.5$ is just the critical value that separates models with negative spatial curvature from models with positive spatial curvature. In any case, a positive value of q, corresponding to a decelerated expansion, was favored over the last years. Therefore it came as a surprise when Riess et al. [30] came to a different conclusion, based on observations of type Ia supernovae in distant galaxies. Presupposing a Friedmann Universe with cosmological constant, they found that their data strongly suggest a negative value of q, corresponding to an accelerated expansion.

It will certainly take a lot more time until we have reliable bounds on the anisotropy of H_2, not to speak of H_3. In this sense, it is too early to draw conclusions about our real universe from the theoretical results obtained in Sections 4 and 5.

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