A general-relativistic Fermat principle for extended light sources and extended receivers

Volker Perlick* and Paolo Piccione†

Abstract

In an arbitrary Lorentzian manifold, we fix a spacelike submanifold P and a timelike submanifold Γ. We interpret P as (the surface of) a light source at a particular instant of time, and we interpret Γ as the history of (the surface of) a receiver. We prove the following version of Fermat's principle. Among all lightlike curves from P to Γ, the lightlike geodesics which are perpendicular to P and spatially perpendicular to Γ are characterized by stationary arrival time. Here, the arrival time is defined with the help of an arbitrary time function on Γ. Moreover, we show that the second variation of the arrival time at a stationary point is characterized by a Morse index theorem.

Key words: variational principle, light rays, Morse theory.

1 INTRODUCTION

For many applications it is useful to characterize light rays by a variational principle rather than by a differential equation. This is true not only in ordinary optics but also in general relativity. In particular, I. Kovner [10] has suggested a general-relativistic variational principle for light rays and he has discussed its relevance in view of applications to gravitational lensing. This variational principle, which can be viewed as a general-relativistic *Fermat principle*, is formulated in the following way. In an arbitrary Lorentzian manifold $(\mathcal{M}, g)$, i.e., in an arbitrary spacetime according to general relativity, one fixes a point p and a timelike curve γ. The point p is to be interpreted as a pointlike light source at a particular instant of time and γ is to be interpreted as the worldline of a pointlike receiver. (There is also a time-reversed version where p is interpreted as a pointlike receiver at a particular instant of time and γ is interpreted as the worldline of a pointlike light source.) Now one considers all lightlike curves

*Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Brazil.
Permanent address: TU Berlin, Sekr. PN 7-1, Institute of Theoretical Physics, 10623 Berlin, Germany. *email*: vper0433@w421zrz.physik.tu-berlin.de

†Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Brazil. *email*: piccione@ime.usp.br

from p to γ, i.e., all possibilities to go from p to γ at the (vacuum) speed of light, and one assigns to each of those curves an "arrival time" with the help of a parametrization for γ. Kovner gave a plausibility argument that, indeed, the lightlike *geodesics* are characterized by stationary arrival time, i.e., that the actual (vacuum) light rays are local extrema or saddle-points of the arrival time. A rigorous mathematical proof of this claim was given in Ref. [13]. The more special versions of Fermat's principle in static, stationary or conformally stationary spacetimes, which are given in several textbooks on general relativity, can be re-obtained easily from this more general version, see Refs. [13] and [14]. Kovner's variational principle was further investigated, both from a physical and from a mathematical point of view, e.g., by L. Bel and J. Martín [2] and in Refs. [15], [5] and [6]. For applications to gravitational lensing, we also refer to the book by P. Schneider, J. Ehlers, and E. Falco [17].

In Kovner's variational principle the light source and the receiver are assumed to be pointlike. In this paper we want to present a more general variational principle where the light source and the receiver may have a spatial extension. To that end we replace the point p with a spacelike submanifold P and we replace the timelike curve γ with a timelike submanifold Γ. (By a timelike submanifold we mean a submanifold such that at each point the tangent space contains a timelike vector.) E.g., P could be a spacelike 2-sphere, to be interpreted as the surface of a light source at a particular instant of time, and Γ could be a timelike 3-manifold, to be interpreted as the history of a screen. Generalizing Kovner's idea, we assign an "arrival time" to each lightlike curve from P to Γ with the help of a time function on Γ, i.e., with the help of a function that slices Γ into spacelike submanifolds. The Fermat principle we are going to prove in Section 3 below says that, among all lightlike curves from P to Γ, the curves of stationary arrival time are exactly the lightlike geodesics which start perpendicularly to P and terminate perpendicularly to a time slice of Γ.

We emphasize that the Fermat principle determines all curves of stationary arrival time and not only those of minimal arrival time. The variational principle itself does not tell whether a lightlike geodesic is a local minimum, a local maximum or a saddle-point of the arrival time functional. It is necessary to calculate a second order variational formula in order to decide which of the three cases is realized. As a matter of fact, the question of whether or not a lightlike geodesic is a local minimum of the arrival time functional depends on the number of focal points along the geodesic and on the geometry (convexity) of the end-manifolds P and Γ. This is the content of a Morse index theorem that will be presented in Section 5 below. In that section we make full use of earlier results obtained by P. Ehrlich and S. Kim [4].

2 ASSUMPTIONS AND NOTATIONS

We consider an arbitrary Lorentzian manifold $(\mathcal{M}, g)$. More precisely, we assume that $\mathcal{M}$ is a finite-dimensional real C^∞ manifold of dimension greater than two whose topology satisfies the second countability axiom and the Hausdorff separation axiom, and we assume that g is a pseudo-Riemannian C^∞ metric of signature $(+, ..., +, -)$ on $\mathcal{M}$. The physically interesting case is, of course, $\dim(\mathcal{M}) = 4$.

We denote by ∇ the Levi–Civita connection and by R the curvature tensor of the Lorentzian metric g.

At each point $p \in \mathcal{M}$, we denote the tangent space to $\mathcal{M}$ by $T_p\mathcal{M}$. We call a linear subspace $\mathcal{W}_p$ of $T_p\mathcal{M}$ *spacelike* if g is positive definite on $\mathcal{W}_p$, *lightlike* if g is positive semidefinite but not positive definite on $\mathcal{W}_p$, and *timelike* otherwise. This implies that $\mathcal{W}_p$ is spacelike if and only if the orthocomplement of $\mathcal{W}_p$ is timelike, and vice versa (cf., e.g., R. Sachs and H. Wu [16], p.20). A vector $v \in T_p\mathcal{M}$ is called *spacelike*, *lightlike* or *timelike* if the linear subspace $\{sv | s \in \mathbb{R}\}$ has the respective property. Equivalently, v is *spacelike* if $g(v, v) > 0$ or $v = 0$, *lightlike* if $g(v, v) = 0$ but $v \neq 0$, and *timelike* if $g(v, v) < 0$. A covector α is called *spacelike*, *lightlike* or *timelike* if the vector v defined by $g(v, \cdot) = \alpha$ has the respective property. Finally, we call a submanifold Σ of $\mathcal{M}$ *spacelike*, *lightlike* or *timelike*, if at all points $p \in \Sigma$ the tangent space $T_p\Sigma$ has the respective property. Note that, according to this definition, a single point is a (zero-dimensional) spacelike submanifold.

To formulate our variational principle we fix a spacelike submanifold P and a timelike submanifold Γ in $\mathcal{M}$. This implies that $0 \leq \dim(P) \leq \dim(\mathcal{M}) - 1$ and $1 \leq \dim(\Gamma) \leq \dim(\mathcal{M})$. Actually, we require $0 \leq \dim(P) \leq \dim(\mathcal{M}) - 2$ and $1 \leq \dim(\Gamma) \leq \dim(\mathcal{M}) - 1$ since in the case $\dim(P) = \dim(\mathcal{M}) - 1$ and/or $\dim(\Gamma) = \dim(\mathcal{M})$ our variational principle would turn out to have no solution. As outlined in the Introduction, P is to be interpreted as (the surface of) a light source at a particular instant of time, whereas Γ is to be interpreted as the history of (the surface of) a receiver. E.g., P could be the surface of a star at a particular instant of time ($\dim(P) = 2$) and Γ could be the history of a screen ($\dim(\Gamma) = 3$) in a 4-dimensional spacetime $(\mathcal{M}, g)$.

As the "trial paths" for our variational principle we want to consider all lightlike curves from P to Γ, and as the functional that is to be extremized we want to consider a kind of "arrival time". To that end we need a *time function* on the manifold Γ, i.e., a C^∞ function $T : \Gamma \to \mathbb{R}$ such that the differential $(dT)_p$ is a timelike covector at all points $p \in \Gamma$. If Γ is one-dimensional, such a time function is just a parametrization for the timelike curve Γ. If $\dim(\Gamma) > 1$, a time function $T : \Gamma \to \mathbb{R}$ exists if and only if Γ, viewed as a Lorentzian manifold in its own right, is stably causal (see, e.g., S. Hawking and G. Ellis [7], p.198). Henceforth we assume that Γ is stably causal, and we choose a particular time function $T : \Gamma \to \mathbb{R}$. Being dT timelike, for all values $c \in \mathbb{R}$ the inverse image $T^{-1}(c)$ is (either empty or) a spacelike codimension-one submanifold of Γ. We will denote these submanifolds by

$$\Gamma_c = T^{-1}(c)\,. \tag{1}$$

Γ_c can be interpreted as the (surface of the) receiver at the time c. Moreover, our time function T determines a unique C^∞ vector field Y on Γ such that

$$dT(Y) = 1 \quad \text{and} \quad Y \perp \Gamma_c\,. \tag{2}$$

The integral curves of this vector field Y can be viewed as the worldlines of the individual points of (the surface of) the receiver.

With $\mathcal{M}$, g, P, Γ and T fixed, we are now ready to formulate our variational problem.

3 THE FERMAT PRINCIPLE

As the "trial paths" for our variational principle we want to consider all piecewise smooth lightlike curves from P to Γ, i.e., all possibilities to go from P to Γ at the (vacuum) speed of light. More precisely, we introduce the space of trial paths in the following way.

Definition 3.1. Let $\mathcal{L}_{P,\Gamma}$ denote the set of all piecewise C^∞ maps $\beta : [0,1] \to \mathcal{M}$ with

(a) $g(\beta', \beta') = 0$;

(b) $\beta(0) \in P$ and $\beta(1) \in \Gamma$;

(c) $g(\beta', U) < 0$.

Here and in the following $\beta' : [0,1] \to T\mathcal{M}$ denotes the tangent field of β. In (c), we have introduced the vector field $U : [0,1] \longrightarrow T\mathcal{M}$ along β which is defined by parallel-transporting the vector $Y_{\beta(1)}$. (Please recall that the vector field Y was introduced through (2)).

Allowing for piecewise smooth curves, rather than just for smooth curves, is of advantage in view of the Morse index theorem to be discussed below. Condition (c) of Definition 3.1 guarantees that $\beta'(1)$ is future-pointing with respect to the time function $T : \Gamma \to I\!R$ and that at the (possible) break points of β the tangent vector field β' does not jump from one half of the light cone to the other.

In the curve space $\mathcal{L}_{P,\Gamma}$ we consider the following kind of variations.

Definition 3.2. For $\beta \in \mathcal{L}_{P,\Gamma}$, an *allowed variation* of β is a map

$$\eta :]-\varepsilon_0, \varepsilon_0[\times [0,1] \longrightarrow \mathcal{M},$$

with some $\varepsilon_0 > 0$, that satisfies the following properties:

(a) there is a subdivision $0 = s_0 < s_1 < ... < s_{N-1} < s_N = 1$ such that η is a C^∞ map on each $]-\varepsilon_0, \varepsilon_0[\times [s_i, s_{i+1}]$;

(b) for all $\varepsilon \in]\varepsilon_0, \varepsilon_0[$ the curve $s \longmapsto \eta(\varepsilon, s)$ is in $\mathcal{L}_{P,\Gamma}$;

(c) $\eta(0, s) = \beta(s)$ for all $s \in [0,1]$.

For $\beta \in \mathcal{L}_{P,\Gamma}$, we define the space $T_\beta \mathcal{L}_{P,\Gamma}$ of *variational vector fields* along β as

$$T_\beta \mathcal{L}_{P,\Gamma} = \Big\{ X \,:\, [0,1] \longrightarrow T\mathcal{M} \;\Big|\; X \text{ piecewise smooth vector field along } \beta, \\ g(\nabla_{\beta'} X, \beta') = 0,\; X(0) \in T_{\beta(0)}P,\; X(1) \in T_{\beta(1)}\Gamma \Big\}. \tag{3}$$

Then the following proposition holds true.

Proposition 3.3. *Let β be any curve in $\mathcal{L}_{P,\Gamma}$. Then every allowed variation η of β defines a variational vector field $X \in T_\beta \mathcal{L}_{P,\Gamma}$ by*

$$X(s) = \eta(\cdot, s)'(0). \tag{4}$$

Conversely, to every $X \in T_\beta \mathcal{L}_{P,\Gamma}$ there is an allowed variation η of β such that (4) *holds.*

Proof. (Sketch) If η is an allowed variation of β, equation (4) clearly defines a piecewise C^∞ vector field along β with $X(0)$ tangent to P and $X(1)$ tangent to Γ. To prove that $g(\nabla_{\beta'} X, \beta') = 0$ we introduce two vector fields ∂_s and ∂_ε along the map η according to

$$\partial_s(\varepsilon, s) = \eta(\varepsilon, \cdot)'(s)\ ,\ \partial_\varepsilon(\varepsilon, s) = \eta(\cdot, s)'(\varepsilon). \tag{5}$$

Since all curves $\eta(\varepsilon, \cdot)$ are lightlike, $g(\partial_s, \partial_s) = 0$. Derivative with respect to ε yields $g(\nabla_{\partial_\varepsilon} \partial_s, \partial_s) = 0$. Since ∇ has vanishing torsion,

$$\nabla_{\partial_\varepsilon} \partial_s = \nabla_{\partial_s} \partial_\varepsilon. \tag{6}$$

Thus

$$g(\nabla_{\partial_s} \partial_\varepsilon, \partial_s) = 0. \tag{7}$$

Evaluating at $\varepsilon = 0$ gives the desired equation.

A proof that, conversely, every $X \in T_\beta \mathcal{L}_{P,\Gamma}$ gives rise to an allowed variation is rather cumbrous and will be omitted here. The method of how to construct the desired variation can be carried over from the proof of Lemma 2 in Ref. [13]. □

Clearly, the set $T_\beta \mathcal{L}_{P,\Gamma}$ has the structure of an infinite dimensional real vector space. As suggested by our notation, the reader may view $T_\beta \mathcal{L}_{P,\Gamma}$ as the tangent space of $\mathcal{L}_{P,\Gamma}$ at the point β. Since we do not establish a differentiable structure on $\mathcal{L}_{P,\Gamma}$, this is meant as a mnemonic only.

The functional we want to extremize is the *arrival time* $\tau : \mathcal{L}_{P,\Gamma} \longrightarrow \mathbb{R}$, defined by

$$\tau(\beta) = T(\beta(1)) \quad \text{for all} \quad \beta \in \mathcal{L}_{P,\Gamma}. \tag{8}$$

Without a differentiable structure on $\mathcal{L}_{P,\Gamma}$ we cannot speak of the derivative of τ. However, for any $\beta \in \mathcal{L}_{P,\Gamma}$ and any allowed variation η of β the map $]-\varepsilon, \varepsilon[\longrightarrow \mathbb{R}$, $\varepsilon \longmapsto \tau(\eta(\varepsilon, \cdot)) = T(\eta(\varepsilon, 1))$ is the composition of two C^∞ maps. By the chain rule, its derivative at $\varepsilon = 0$ is given by

$$\frac{d}{d\varepsilon} \tau(\eta(\varepsilon, \cdot)) \Big|_{\varepsilon=0} = (dT)_{\beta(1)}(X(1)) \tag{9}$$

where X is defined through (4). This suggests to view the linear map $(d\tau)_\beta : T_\beta \mathcal{L}_{P,\Gamma} \longrightarrow I\!R$ which is defined by

$$(d\tau)_\beta(X) = (dT)_{\beta(1)}(X(1)) \tag{10}$$

as the differential of τ at the point β. (Again, this is meant as a mnemonic only.) Although we have no differentiable structure on $\mathcal{L}_{P,\Gamma}$, we can now define what we mean by a stationary point of τ.

Definition 3.4. A curve $\beta \in \mathcal{L}_{P,\Gamma}$ is a *stationary point* of τ if

$$\frac{d}{d\varepsilon}\tau(\eta(\varepsilon,\cdot))\Big|_{\varepsilon=0} = 0$$

for all allowed variations η of β. Equivalently, β is a stationary point of τ if $(d\tau)_\beta(X) = 0$ for all $X \in T_\beta \mathcal{L}_{P,\Gamma}$.

Here we make use of the above-mentioned fact that every $X \in T_\beta \mathcal{L}_{P,\Gamma}$ can be written in the form of equation (4) with an allowed variation η of β. The variational problem we want to solve is to determine the stationary points of τ. For that purpose we need the following representation of the map $(d\tau)_\beta$.

Proposition 3.5. *For any $\beta \in \mathcal{L}_{P,\Gamma}$, the map $(d\tau)_\beta$ defined by* (10) *admits the following representation:*

$$\begin{aligned}(d\tau)_\beta(X) \quad &= \quad \frac{g(\beta'(0), X(0))}{g(\beta'(0), U(0))} - \frac{g(\beta'(1), X(1)_\perp)}{g(\beta'(1), U(1))} - \\ -\sum_{i=1}^{N}\Big[\frac{g(\beta', X)}{g(\beta', U)}\Big](s_i) \quad &+ \quad \int_0^1 g\Big(\nabla_{\beta'}\frac{\beta'}{g(\beta', U)},\ X\Big)ds.\end{aligned} \tag{11}$$

for all $X \in T_\beta \mathcal{L}_{P,\Gamma}$. Here U denotes the vector field along β defined by parallel-transporting the vector $Y_{\beta(1)}$, as in Definition 3.1. *$X(1)_\perp$ is the component of $X(1)$ perpendicular to $Y(1)$. The $s_i \in \,]0,1[\,$, $i = 1, ..., N$ are those parameter values at which β' is discontinuous, and the square bracket denotes the jump of the respective function, $[F](s_i) = F(s_i + 0) - F(s_i - 0)$.*

Proof. Let η be any allowed variation of β. Again, we use the vector fields ∂_s and ∂_ε along the map η defined by (5). Moreover, we denote by $U_\varepsilon : [0,1] \longrightarrow T\mathcal{M}$ the vector field defined along $\eta(\varepsilon,\cdot)$ by parallel-transporting the vector $Y_{\eta(\varepsilon,1)}$. Using the subdivision $0 = s_0 < s_1 < ... < s_{N-1} < s_N = 1$ of Definition 3.2 (a), the fundamental theorem of calculus yields

$$\frac{g(\partial_s, \partial_\varepsilon)}{g(\partial_s, U_\varepsilon)}\Big|_{(\varepsilon, s_{i+1}-0)} - \frac{g(\partial_s, \partial_\varepsilon)}{g(\partial_s, U_\varepsilon)}\Big|_{(\varepsilon, s_i+0)} = \int_{s_i}^{s_{i+1}} \frac{\partial}{\partial s}\frac{g(\partial_s, \partial_\varepsilon)}{g(\partial_s, U_\varepsilon)} ds. \tag{12}$$

Summation over i from 0 to $N-1$ results in

$$\begin{aligned}\frac{g(\partial_s, \partial_\varepsilon)}{g(\partial_s, U_\varepsilon)}\Big|_{(\varepsilon,1)} \quad &- \quad \frac{g(\partial_s, \partial_\varepsilon)}{g(\partial_s, U_\varepsilon)}\Big|_{(\varepsilon,0)} + \sum_{i=1}^{N-1}\Big[\frac{g(\partial_s, \partial_\varepsilon)}{g(\partial_s, U_\varepsilon)}\Big]\Big|_{(\varepsilon, s_i)} = \\ &= \quad \int_0^1 g\Big(\nabla_{\partial_s}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, \partial_\varepsilon\Big)ds.\end{aligned} \tag{13}$$

In the last term we have used equation (7). Now we decompose the vector $\partial_\varepsilon(\varepsilon,1)$ into a component parallel and a component orthogonal to the timelike vector $U_\varepsilon(1) = Y_{\eta(\varepsilon,1)}$:

$$\partial_\varepsilon(\varepsilon,1) = a(\varepsilon)U_\varepsilon(1) + (\partial_\varepsilon)_\perp(\varepsilon,1). \tag{14}$$

Application of the one-form dT to this equation shows that

$$a(\varepsilon) = (dT)_{\eta(\varepsilon,1)}(\partial_\varepsilon(\varepsilon,1)) = \frac{d}{d\varepsilon}\tau(\eta(\varepsilon,\cdot)). \tag{15}$$

Inserting (14) and (15) into (13) yields

$$\begin{aligned}\frac{d}{d\varepsilon}\tau(\eta(\varepsilon,\cdot)) = \frac{g(\partial_s,\partial_\varepsilon)}{g(\partial_s,U_\varepsilon)}\Big|_{(\varepsilon,0)} - \frac{g(\partial_s,(\partial_\varepsilon)_\perp)}{g(\partial_s,U_\varepsilon)}\Big|_{(\varepsilon,1)} - \\ - \sum_{i=1}^{N-1}\Big[\frac{g(\partial_s,\partial_\varepsilon)}{g(\partial_s,U_\varepsilon)}\Big]\Big|_{(\varepsilon,s_i)} + \int_0^1 g\Big(\nabla_{\partial_s}\frac{\partial_s}{g(\partial_s,U_\varepsilon)},\partial_\varepsilon\Big)ds.\end{aligned} \tag{16}$$

Evaluation at $\varepsilon = 0$ gives the desired result for the arbitrary element $X = \partial_\varepsilon(0,\cdot) \in T_\beta\mathcal{L}_{P,\Gamma}$. □

We are now ready to prove our variational principle.

Theorem 3.6. *Let β be a curve in $\mathcal{L}_{P,\Gamma}$. Then β is a stationary point of the arrival time functional τ if and only if $\beta\circ\phi^{-1}$ is an affinely parametrized geodesic, $\beta'(0)$ is orthogonal to $T_{\beta(0)}P$ and $\beta'(1)$ is orthogonal to $T_{\beta(1)}\Gamma_{\tau(\beta)}$. Here $\phi : [0,1] \longrightarrow [0,1]$ denotes the piecewise C^∞ diffeomorphism defined by*

$$\phi(s) = \frac{\int_0^s g(\beta',U)d\tilde{s}}{\int_0^1 g(\beta',U)d\tilde{s}}. \tag{17}$$

Proof. First we prove the "if" part which is easy with the help of Proposition 3.5. By assumption, $\beta\circ\phi^{-1}$ is an affinely parametrized geodesic and, thus, a smooth curve. This implies that $\frac{\beta'}{g(\beta',U)}$ is everywhere continuous and that $\nabla_{\beta'}\frac{\beta'}{g(\beta',U)} = 0$. Hence, the last two terms in (11) vanish. Since $\beta'(0)$ is orthogonal to $T_{\beta(0)}P$ and $\beta'(1)$ is orthogonal to $T_{\beta(1)}\Gamma_{\tau(\beta)}$, the first two terms on the right-hand side of (11) also vanish, so $(d\tau)_\beta(X) = 0$ for all $X \in T_\beta\mathcal{L}_{P,\Gamma}$.

Now we prove the "only if" part. Let $0 = s_0 < s_1 < ... < s_{N-1} < s_N = 1$ be such that β is a C^∞ map on each interval $[s_i, s_{i+1}]$. Choose any continuous vector field $V : [0,1] \longrightarrow T\mathcal{M}$ along β which is a C^∞ map on each interval $[s_i, s_{i+1}]$ with $V(0)$ tangent to P and $V(1)$ tangent to $\Gamma_{\tau(\beta)}$. Then it is readily verified that the vector field $X = V - fU$ belongs to $T_\beta\mathcal{L}_{P,\Gamma}$, where

$$f(s) = \int_0^s \frac{g(\nabla_{\beta'}V,\beta')}{g(\beta',U)}d\tilde{s}. \tag{18}$$

By Proposition 3.5, our assumption that β is a critical point of τ implies

$$0 = \frac{g(\beta'(0), V(0))}{g(\beta'(0), U(0))} \quad - \quad \frac{g(\beta'(1), V(1))}{g(\beta'(1), U(1))} -$$
$$- \sum_{i=1}^{N} \Big[\frac{g(\beta', V)}{g(\beta', U)} \Big] (s_i) \quad + \quad \int_0^1 g\Big(\nabla_{\beta'} \frac{\beta'}{g(\beta', U)}, \; V \Big) ds. \tag{19}$$

On the right-hand side we have used that fU gives no contribution (i) to the first term since $f(0) = 0$, (ii) to the second term since $X(1)_\perp = V(1)$, (iii) to the third term since f is continuous and (iv) to the fourth term since $\nabla_{\beta'} U = 0$. If we specialize to vector fields V which are zero outside of one interval $]s_i, s_{i+1}[$, equation (19) reduces to

$$0 = \int_{s_i}^{s_{i+1}} g\big(\nabla_{\beta'} \frac{\beta'}{g(\beta', U)}, \; V \big) ds. \tag{20}$$

Owing to the fundamental lemma of variational calculus, this implies that

$$\nabla_{\beta'} \frac{\beta'}{g(\beta', U)} = 0, \tag{21}$$

i.e., that $\beta \circ \phi^{-1}$ is an affinely parametrized geodesic, on each interval $[s_i, s_{i+1}]$. On the other hand, we can specialize to vector fields V which take arbitrary values at a particular s_j, $1 \leq j \leq N-1$, and are zero at all the other s_i's. For such vector fields V, owing to (21), equation (19) simplifies to

$$\Big[\frac{g(\beta', V)}{g(\beta', U)} \Big] (s_j) = 0. \tag{22}$$

Hence, $\frac{\beta'}{g(\beta', U)}$ has no discontinuities. Together with (21) this implies that $\beta \circ \phi^{-1}$ is, indeed, an (unbroken) affinely parametrized geodesic on all of $[0,1]$. To prove the boundary conditions, we specialize to vector fields V with arbitrary initial values $V(0)$ tangent to P and $V(1) = 0$. Then (19) takes the form $g(\beta'(0), V(0)) = 0$. This shows that $\beta'(0)$ must be orthogonal to $T_{\beta(0)}P$. Similarly, we can specialize to vector fields V with arbitrary end values $V(1)$ tangent to $\Gamma_{\tau(\beta)}$ and $V(0) = 0$. Then (19) reduces to $g(\beta'(1), V(1)) = 0$. This shows that $\beta'(1)$ must be orthogonal to $T_{\beta(1)} \Gamma_{\tau(\beta)}$. □

If we specialize Theorem 3.6 to the case $\dim(P) = 0$ and $\dim(\Gamma) = 1$ we re-obtain, apart from a mathematical subtlety, the version of Fermat's principle proven in Ref. [13]. This mathematical subtlety is in the fact that in Ref. [13] only C^∞ curves were considered as trial paths whereas here we allow for piecewise C^∞ curves.

4 THE SECOND ORDER VARIATIONAL FORMULA

Now we want to inquire whether a stationary point β is a minimum, a maximum or a saddle-point of τ. To that end we need a second order variational formula for our variational principle. We begin with the following observation.

Proposition 4.1. *If $\beta \in \mathcal{L}_{P,\Gamma}$ is a stationary point of τ, the tangent space* (3) *takes the following form.*

$$T_\beta \mathcal{L}_{P,\Gamma} = \Big\{ X : [0,1] \longrightarrow T\mathcal{M} \;\Big|\; X \text{ piecewise smooth vector field along } \beta, \quad g(\beta', X) = 0, \quad X(0) \in T_{\beta(0)}P, \; X(1) \in T_{\beta(1)}\Gamma_{\tau(\beta)} \Big\} . \tag{23}$$

Proof. We have to show that, for a variational vector field X along a stationary point, the condition $X(1) \in T_{\beta(1)}\Gamma$ is equivalent to $X(1) \in T_{\beta(1)}\Gamma_{\tau(\beta)}$, and the condition $g(\beta', \nabla_{\beta'} X) = 0$ is equivalent to $g(\beta', X) = 0$. The first claim follows from (14) and (15) evaluated at $\varepsilon = 0$.

To prove the second claim, we recall that, by Theorem 3.6, our assumption implies $\nabla_{\beta'} \frac{\beta'}{g(\beta', U)} = 0$. Thus, $g(\beta', \nabla_{\beta'} X) = 0$ is equivalent to $\frac{d}{ds} \frac{g(\beta', X)}{g(\beta', U)} = 0$, i.e., to $\frac{g(\beta', X)}{g(\beta', U)} = const$. The latter can hold only if $g(\beta', X) = 0$ since $g(\beta'(0), X(0)) = 0$. □

As a preparation for the following we have to recall that, for any submanifold Σ of our Lorentzian manifold $\mathcal{M}$ whose tangent bundle $T\Sigma$ contains no lightlike vectors, the *second fundamental form* S^Σ (also known as the *shape tensor* of Σ) can be defined in analogy to the Riemannian case. Namely, for each $p \in \Sigma$ and each vector $n \in T_p\mathcal{M}$ which is perpendicular to $T_p\Sigma$ the second fundamental form of Σ in the direction of n is the bilinear form $S_n^\Sigma : T_p\Sigma \times T_p\Sigma \longrightarrow I\!R$ defined by

$$S_n^\Sigma(v_1, v_2) = g(n, \nabla_{v_1} V_2) \tag{24}$$

where V_2 is any C^∞ vector field on Σ which takes the value v_2 at p. As in Riemannian geometry, one can show that S_n^Σ is, indeed, well defined (i.e., independent of the extension V_2 of v_2) and symmetric. For details we refer the reader to J. Beem, P. Ehrlich and K. Easley [1] and to B. O'Neill [12].

We are now ready to prove the desired second variational formula.

Proposition 4.2. *Let $\beta \in \mathcal{L}_{P,\Gamma}$ be a stationary point of τ. Let η be an allowed variation of β with pertaining variational vector field $X \in T_\beta \mathcal{L}_{P,\Gamma}$ defined through* (4)*. Then*

$$\frac{\mathrm{d}^2}{d\varepsilon^2} \tau(\eta(\varepsilon, \cdot)) \Big|_{\varepsilon=0} = H_\beta^\tau(X, X) \tag{25}$$

where the bilinear form $H_\beta^\tau : T_\beta \mathcal{L}_{P,\Gamma} \times T_\beta \mathcal{L}_{P,\Gamma} \longrightarrow I\!R$ *is defined by*

$$\begin{aligned} H_\beta^\tau(X_1, X_2) &= \frac{S^P_{\beta'(0)}(X_1(0), X_2(0))}{g(\beta'(0), U(0))} - \frac{S^{\Gamma_{\tau(\beta)}}_{\beta'(1)}(X_1(1), X_2(1))}{g(\beta'(1), U(1))} + \\ &+ \int_0^1 \frac{\big(g(R(X_1, \beta', \beta'), X_2) - g(\nabla_{\beta'} X_1, \nabla_{\beta'} X_2)\big)}{g(\beta', U)} ds. \end{aligned} \tag{26}$$

Proof. As before, let ∂_s and ∂_ε denote the two vector fields along η defined by (5) and let U_ε be the vector field along $\eta(\varepsilon, \cdot)$ defined by parallel-transporting the vector $Y_{\eta(\varepsilon,1)}$. Then we know from the proof of Proposition 3.5 that the first derivative of the map $\varepsilon \longmapsto \tau(\eta(\varepsilon, \cdot))$ is given by equation (16). If we apply the operator $d/d\varepsilon$ another time we get

$$\begin{aligned} \frac{d^2}{d\varepsilon^2}\tau(\eta(\varepsilon,\cdot)) &= \Big(\frac{g(\partial_s, \nabla_{\partial_\varepsilon}\partial_\varepsilon)}{g(\partial_s, U_\varepsilon)} + g\big(\nabla_{\partial_\varepsilon}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, \partial_\varepsilon\big)\Big)\Big|_{(\varepsilon,0)} - \\ &-\Big(\frac{g(\partial_s, \nabla_{\partial_\varepsilon}(\partial_\varepsilon)_\perp)}{g(\partial_s, U_\varepsilon)} + g\big(\nabla_{\partial_\varepsilon}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, (\partial_\varepsilon)_\perp\big)\Big)\Big|_{(\varepsilon,1)} - \\ &- \sum_{i=1}^{N-1}\Big[\frac{g(\partial_s, \nabla_{\partial_\varepsilon}\partial_\varepsilon)}{g(\partial_s, U_\varepsilon)} + g\big(\nabla_{\partial_\varepsilon}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, \partial_\varepsilon\big)\Big](s_i) + \\ &+ \int_0^1 \Big(g\big(\nabla_{\partial_\varepsilon}\nabla_{\partial_s}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, \partial_\varepsilon\big) + g\big(\nabla_{\partial_s}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, \nabla_{\partial_\varepsilon}\partial_\varepsilon\big)\Big)ds. \end{aligned} \tag{27}$$

The first term under the integral can be rewritten with the help of the curvature tensor R as

$$\begin{aligned} g\Big(\nabla_{\partial_\varepsilon}\nabla_{\partial_s}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, \partial_\varepsilon\Big) &= \\ = g\Big(R(\partial_\varepsilon, \partial_s, \frac{\partial_s}{g(\partial_s, U_\varepsilon)}) &+ \nabla_{\partial_s}\nabla_{\partial_\varepsilon}\frac{\partial_s}{g(\partial_s, U_\varepsilon)},\ \partial_\varepsilon\Big) = \\ = g\Big(R(\partial_\varepsilon, \partial_s, \frac{\partial_s}{g(\partial_s, U_\varepsilon)}), \partial_\varepsilon\Big) &+ \frac{\partial}{\partial s}g\Big(\nabla_{\partial_\varepsilon}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, \partial_\varepsilon\Big) - \\ &- g\Big(\nabla_{\partial_\varepsilon}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, \nabla_{\partial_s}\partial_\varepsilon\Big). \end{aligned} \tag{28}$$

Upon inserting this expression into (27), one term can be integrated over each interval $[s_i, s_{i+1}]$ and cancels with part of the boundary terms. With the help of (6) and (7), (27) takes the following form.

$$\begin{aligned} \frac{d^2}{d\varepsilon^2}\ \tau(\eta(\varepsilon,\cdot)) &= \frac{g(\partial_s, \nabla_{\partial_\varepsilon}\partial_\varepsilon)}{g(\partial_s, U_\varepsilon)}\Big|_{(\varepsilon,0)} - \frac{g(\partial_s, \nabla_{\partial_\varepsilon}(\partial_\varepsilon)_\perp)}{g(\partial_s, U_\varepsilon)}\Big|_{(\varepsilon,1)} - \\ &-g\Big(\nabla_{\partial_\varepsilon}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, (\partial_\varepsilon)_\perp - \partial_\varepsilon\Big)\Big|_{(\varepsilon,1)} - \sum_{i=1}^{N-1}\Big[\frac{g(\partial_s, \nabla_{\partial_\varepsilon}\partial_\varepsilon)}{g(\partial_s, U_\varepsilon)}\Big](s_i) + \\ &+ \int_0^1 \Big(\frac{g(R(\partial_\varepsilon, \partial_s, \partial_s), \partial_\varepsilon)}{g(\partial_s, U_\varepsilon)} \frac{g(\nabla_{\partial_s}\partial_\varepsilon, \nabla_{\partial_s}\partial_\varepsilon))}{g(\partial_s, U_\varepsilon)} + g\big(\nabla_{\partial_s}\frac{\partial_s}{g(\partial_s, U_\varepsilon)}, \nabla_{\partial_\varepsilon}\partial_\varepsilon\big)\Big)ds. \end{aligned} \tag{29}$$

Now we set ε equal to zero and we make use of the fact that β is a stationary point of τ. By Theorem 3.6, the latter condition implies that

$$\nabla_{\beta'} \frac{\beta'}{g(\beta', U)} = 0,$$

so the last term under the integral vanishes; and it implies that $\frac{\beta'}{g(\beta',U)}$ is continuous, so the third term on the right-hand side vanishes. Finally, Proposition 4.1 allows to replace $X(1)_\perp$ with $X(1)$, so we get the desired formula. □

The bilinear form H^τ_β defined by (26) can be interpreted as the *Hessian* of τ at the stationary point β. Please note that H^τ_β is, indeed, symmetric. This follows from the above-mentioned fact that the second fundamental forms $S^P_{\beta(0)}$ and $S^{\Gamma_{\tau(\beta)}}_{\beta(1)}$ are symmetric and from the well-known symmetry properties of the curvature tensor.

5 THE MORSE INDEX THEOREM

We want to use the second order variational formula of Proposition 4.2 to inquire whether a stationary point β is a minimum, a maximum or a saddle-point of τ. Throughout this section we assume that $\beta \in \mathcal{L}_{P,\Gamma}$ is an affinely parametrized geodesic. This is no restriction of generality since, by Theorem 3.6, every stationary point of τ is a piecewise C^∞ reparametrization of an affinely parametrized geodesic and τ is obviously invariant under such reparametrizations. If β is an affinely parametrized geodesic, the function $g(\beta', U)$ is obviously constant, so it can be replaced with its value at $s = 1$. Then the Hessian (26) simplifies to

$$H^\tau_\beta(X_1, X_2) = g(\beta'(1), Y_{\beta(1)})^{-1} I_\beta(X_1, X_2) \tag{30}$$

where

$$\begin{aligned} I_\beta(X_1, X_2) \;&=\; S^P_{\beta'(0)}(X_1(0), X_2(0)) - S^{\Gamma_{\tau(\beta)}}_{\beta'(1)}(X_1(1), X_2(1)) + \\ &+\; \int_0^1 \big(g(R(X_1, \beta', \beta'), X_2) - g(\nabla_{\beta'} X_1, \nabla_{\beta'} X_2)\big) ds. \end{aligned} \tag{31}$$

The bilinear form $I_\beta : T_\beta\mathcal{L}_{P,\Gamma} \times T_\beta\mathcal{L}_{P,\Gamma} \longrightarrow I\!\!R$ is known as the *index form* of β, where $T_\beta\mathcal{L}_{P,\Gamma}$ is given by (23). The properties of the index form I_β have been investigated by P. Ehrlich and S. Kim [4] in connection with a Morse theory for lightlike geodesics between two spacelike end-manifolds K_1 and K_2. (In our case, $K_1 = P$ and $K_2 = \Gamma_{\tau(\beta)}$.) In what follows we review the results of Ehrlich and Kim on the index form I_β. We rewrite them as results on our Hessian H^τ_β, using the fact that, by (30), these two bilinear forms differ only by a constant negative factor.

First we observe that $H^\tau_\beta(X, \cdot)$ vanishes whenever X is a multiple of β'. This reflects the fact that our arrival time functional τ is invariant under reparametrizations. As an immediate consequence, β cannot be a strict local minimum (or a strict local maximum) of τ with respect to all allowed variations. It is therefore recommendable to restrict to *non-trivial* variations, i.e., to variations whose variational vector fields are not just multiples of β'. To work this out in technical terms, we consider two elements of $T_\beta \mathcal{L}_{P,\Gamma}$ as *equivalent* if they differ by a multiple of β' and we denote the corresponding quotient space by $\tilde{T}_\beta \mathcal{L}_{P,\Gamma}$. We define bilinear forms $\tilde{H}^\tau_\beta,\ \tilde{I}_\beta : \tilde{T}_\beta \mathcal{L}_{P,\Gamma} \times \tilde{T}_\beta \mathcal{L}_{P,\Gamma} \longrightarrow \mathbb{R}$ by

$$\tilde{H}^\tau_\beta([X],[X]) = H^\tau_\beta(X,X)\,,\ \ \tilde{I}_\beta([X],[X]) = I_\beta(X,X) \tag{32}$$

where X is any element of the equivalence class $[X]$. Contrary to H^τ_β, the bilinear form $\tilde{H}^\tau_\beta$ has a chance to be non-degenerate. If this is the case, the index of $\tilde{H}^\tau_\beta$ is called the *Morse index* of β. (Here the "index" of a bilinear form is defined as the maximal dimension of a subspace on which the form is negative definite.) Clearly, in the case that $\tilde{H}^\tau_\beta$ is non-degenerate, the Morse index of β is zero if and only if β is a strict local minimum of τ with respect to all non-trivial variations.

Please note that the Morse index of β is the maximal dimension of a subspace on which $\tilde{I}_\beta$ is *positive* definite, thus our terminology is in agreement with the terminology of P. Ehrlich and S. Kim [4]. They have shown that the Morse index of β is determined by the number of focal points along β and by a quantity that measures the "convexity" of $K_1 = P$ and $K_2 = \Gamma_{\tau(\beta)}$. In order to review this result we need the following definition.

Definition 5.1. Let $\beta \in \mathcal{L}_{P,\Gamma}$ be an affinely parametrized geodesic. A C^∞ vector field J along β is called a *P-Jacobi field* if

(a) $\nabla_{\beta'} \nabla_{\beta'} J + R(J, \beta', \beta') = 0$;

(b) $g(J, \beta') = 0$;

(c) $J(0) \in T_{\beta(0)} P$;

(d) $g(\nabla_{\beta'(0)} J, v) + S^P_{\beta'(0)}(J(0), v) = 0$ for all $v \in T_{\beta(0)} P$.

By (a) "the arrow-head of J traces out an infinitesimally close neighboring geodesic"; by (b) this neighboring geodesic is spatially separated from β and, again, lightlike; by (c) it starts, again, on P; by (d) it is, again, perpendicular to P. In the special case that P is a point, (c) requires $J(0) = 0$ and (d) is no condition at all. Please note that, in general, a P-Jacobi field does not belong to $T_\beta \mathcal{L}_{P,\Gamma}$ since it need not satisfy any boundary condition at $s = 1$. We call two P-Jacobi fields along β *equivalent* if they differ by a multiple of β'. The corresponding equivalence classes will be called *P-Jacobi classes*. For $s \in]0,1]$, we denote by $\mathcal{J}_s$ the set of all P-Jacobi classes $[J]$ along β with $J(s)$ parallel to $\beta'(s)$ (for one and, thus, for any representative $J \in [J]$). Clearly, $\mathcal{J}_s$ is a real vector space. If $\dim(\mathcal{J}_s) \neq 0$, we call $\beta(s)$ a *P-focal point* and we call $\dim(\mathcal{J}_s)$ its *multiplicity*. In particular in the case that P is a point, a P-focal point is also called a *point conjugate to P*.

Moreover, we denote by $\mathcal{R}$ the set of all P-Jacobi classes $[J]$ along β such that $J(1)$ is the linear combination of a vector tangent to $\Gamma_{\tau(\beta)}$ and a multiple of $\beta'(1)$ (for one and, thus, for any representative $J \in [J]$). On the real vector space $\mathcal{R}$ we define a bilinear form $\mathcal{S} : \mathcal{R} \times \mathcal{R} \longrightarrow I\!R$ by

$$\mathcal{S}([J_1],[J_2]) = g(\nabla_{\beta'(1)} J_1, J_2) + S^{\Gamma_{\tau(\beta)}}_{\beta'(1)}(J_1(1), J_2(1)) \tag{33}$$

for $J_1 \in [J_1]$ and $J_2 \in [J_2]$. A quick calculation shows that $\mathcal{S}$ is, indeed, well-defined and symmetric.

We are now ready to formulate the desired Morse index theorem.

Theorem 5.2. (Ehrlich and Kim [4]) *Let $\beta \in \mathcal{L}_{P,\Gamma}$ be an affinely parametrized geodesic and assume that $\beta(1)$ is not a P-focal point along β. Then $\tilde{H}^{\tau}_{\beta}$ is non-degenerate and the Morse index of β equals the number of P-focal points along β counted with multiplicities plus the index of the bilinear form $\mathcal{S}$ defined by* (33)*,*

$$\mathrm{Index}(\tilde{H}^{\tau}_{\beta}) = \sum_{s \in]0,1[} \dim(\mathcal{J}_s) + \mathrm{Index}(\mathcal{S}). \tag{34}$$

Here $\mathrm{Index}(\Phi)$ *denotes the maximal dimension of a subspace on which the bilinear form Φ is negative definite.*

A proof of this theorem was given by P. Ehrlich and S. Kim, see Ref. [4] and, for some partial results necessary, Ref. [3].

If $\Gamma_{\tau(\beta)}$ is a point (i.e., if $\dim(\Gamma) = 1$), the bilinear form $\mathcal{S}$ is defined on the zero-space $\mathcal{R} = \{0\}$, hence $\mathrm{Index}(\mathcal{S}) = 0$. In that case, by Theorem 5.2, the Morse index of β equals the number of P-focal points along β counted with multiplicities. In particular, β is a strict local minimum of τ with respect to non-trivial variations if and only if β is free of P-focal points. A similar argument applies if P is a point since for the Morse index theorem the roles of P and $\Gamma_{\tau(\beta)}$ are interchangeable (although for Fermat's principle they are, of course, not). Hence, in that case the Morse index of β is given by the number of $\big(\Gamma_{\tau(\beta)}\big)$-focal points counted with multiplicities.

This shows that the bilinear form $\mathcal{S}$ is relevant only in the case that both the light source and the receiver have a spatial extension. To get a geometric idea of the role of $\mathcal{S}$ it is helpful to consider the following example. If β is a lightlike geodesic in Minkowski space that starts from a spacelike 2-sphere P with an outwards-pointing initial vector, it is clear that there are no P-focal points on β, because of the flatness of the Minkowski metric. So, in that case, the Morse index of β equals the Index of $\mathcal{S}$ for any possible choice of Γ. On the other hand, considering that the functional τ measures the length of the *spatial part* of β, it is geometrically evident that such a β fails to be a local minimum for τ only in the case that $\Gamma_{\tau(\beta)}$ is curved in the direction towards P with a sufficiently small curvature radius. This example illustrates that $\mathrm{Index}(\mathcal{S})$ may be viewed as a measure for the "convexity" of the end-manifolds.

By condition (a) of Definition 5.1, the real vector spaces $\mathcal{J}_s$ and $\mathcal{R}$ are finite-dimensional. Moreover, there are only finitely many P-focal points on the compact interval $[0, 1]$. (For a proof we refer to P. Ehrlich and S. Kim [3].) Hence, Theorem 5.2 implies that the Morse index of β is always finite. In other words, β cannot be a local maximum of τ. This is geometrically quite evident since it is always possible to construct allowed variations where the neighboring curves have more "wiggles" than β.

Finally, it is interesting to specialize the results of this paper to the conformally stationary case, i.e., to the case that the spacetime admits a timelike conformal Killing vector field W. If some global regularity assumptions are satisfied, the lightlike geodesic equation in the n-dimensional spacetime $\mathcal{M}$ can then be reduced to an equation in an $(n-1)$-dimensional space $\hat{\mathcal{M}}$, see, e.g., Ref. [14]. In the non-rotating (i.e., conformally static) case, the projected light rays are just the geodesics of a Riemannian metric $\hat{g}$ which was called the *Fermat metric* in Ref. [14]. Otherwise they are modified by a kind of Coriolis force and can then be interpreted as the geodesics of a *pseudo-Finslerian* structure, see Ref. [11].

If we assume, in addition, that the timelike manifold Γ is invariant under the flow of W and that the time function T is chosen such that $dT(W) = 1$ on Γ, the Fermat principle of Theorem 3.6 can be reduced to a variational principle for curves in $\hat{\mathcal{M}}$. In the non-rotating case, this reduced variational principle is just a Riemannian geodesic problem, viz., to determine curves of stationary $\hat{g}$-arc length between two end-manifolds. Correspondingly, Theorem 5.2 reduces to the Morse index theorem for Riemannian geodesics between end-manifolds in the version of D. Kalish [8], [9].

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