Gravitational lensing in asymptotically simple and empty spacetimes

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Abstract. An asymptotically simple and empty spacetime provides a reasonable model for a transparent gravitational lens deflector as long as cosmological aspects can be ignored. Here we show that in this situation a lens map for light sources at infinity can be introduced, and we indicate that, for a generic gravitational lens situation in an asymptotically simple and empty spacetime, the number of images is finite and odd.

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1 Introduction

If spacetime is modeled as a time-oriented Lorentzian manifold, a *gravitational lens* situation is characterized by the fact that a point (observation event) can be joined to a timelike curve (worldline of light source) by two or more past-pointing geodesics (light rays). In realistic situations gravitational lensing is brought about by the bending of light rays in the gravitational field of a heavy mass (deflector).

Most theoretical work on gravitational lensing is done not in a Lorentzian geometry setting but rather in a quasi-Newtonian approximation formalism, see, e.g., Schneider, Ehlers and Falco [1]. This approximation formalim is used not only for calculating particular models but also for proving general features of gravitational lensing. As to the latter aspect, it is an interesting task to investigate to what extent the results depend on the approximations involved. It is the goal of this article to demonstrate that several general features of gravitational lensing, known from the quasi-Newtonian approximation formalism, can be nicely reformulated in a Lorentzian geometry setting if one restricts to asymptotically simple and empty spacetimes. Roughly speaking, this covers all gravitational lensing situations where the deflector is transparent and isolated. Among other things, we state a theorem that, under these assumptions, a gravitational lens produces an odd number of images. In the quasi-Newtonian approximation formalism, such an odd number theorem was proven by Burke [2]. An attempt to reformulate this result in a Lorentzian manifold setting was brought forward by McKenzie [3]. In comparison to the work of McKenzie the present approach has the advantage that it applies to a class of spacetimes which is physically well understood.

2 Asymptotically simple and empty spacetimes

By a *spacetime* we mean a time-oriented 4-dimensional Lorentzian manifold (M, g) of class C^∞. We assume that the reader is familiar with basic causality notions such as strong causality and global hyperbolicity, see, e.g., Hawking and Ellis [4]. We use the following standard definition, cf., e.g., Hawking and Ellis [4], p. 222.

Definition 2.1. A spacetime (M, g) is called *asymptotically simple* if there is a strongly causal spacetime $(\tilde{M}, \tilde{g})$ with the following properties.

(a) M is an open submanifold of $\tilde{M}$ with a non-empty boundary ∂M.

(b) There is a C^∞ function $\Omega : \tilde{M} \longrightarrow I\!R$ such that $M = \{p \in \tilde{M} | \Omega(p) > 0\}$, $\partial M = \{p \in \tilde{M} | \Omega(p) = 0\}$, and the equation $\tilde{g} = \Omega^2 g$ holds on M. Moreover, $d\Omega$ has no zeros on ∂M.

(c) Every inextendible lightlike geodesic in M has two endpoints on ∂M.

(M, g) is called *asymptotically simple and empty* if, in addition,

(d) there is a neighborhood U of ∂M in $\tilde{M}$ such that the Ricci tensor of g vanishes on $U \cap M$.

Asymptotically simple and empty spacetimes are good models for isolated gravitating bodies. Condition (d) of Definition 2.1. is a way of saying that, sufficiently far away from the gravitating body under consideration, Einstein's vacuum field equation is satisfied. This is a reasonable model for a gravitational lens deflector as long as cosmological aspects can be ignored.

Conditions (b) and (c) of Definition 2.1. imply that in an asymptotically simple spacetime all lightlike geodesics are complete. Hence, the elements of ∂M are to be interpreted as points at infinity which can be reached along light rays. In view of gravitational lensing it is then clear that condition (c) excludes non-transparent deflectors.

In the next section we want to introduce the notion of "light sources at infinity". To that end we have to recall the following well-known facts about asymptotically simple and empty spacetimes, cf. Hawking and Ellis [4], p. 222. Conditions (a)–(d) of Definition 2.1. imply that ∂M is a 3-dimensional $\tilde{g}$-lightlike submanifold of $\tilde{M}$ which has exactly two connected components: $\mathcal{J}^+$ (pronounced "scri plus") where future-pointing g-lightlike geodesics terminate and $\mathcal{J}^-$ where past-pointing g-lightlike geodesics terminate. $\mathcal{J}^+$ and $\mathcal{J}^-$ are ruled by the integral curves of the vector field Z defined by the equation $d\Omega = \tilde{g}(Z, \cdot)$. Those integral curves are lightlike geodesics of the metric $\tilde{g}$ (up to parametrization); they are usually called the *generators* of $\mathcal{J}^\pm$.

Moreover, we shall need the following important theorem which determines the global structure of asymptotically simple and empty spacetimes. It is eessentially due to Geroch [5]. A subtlety overlooked in the original proof of Geroch [5] was clarified by Newman and Clarke [6].

Theorem 2.2. An asymptotically simple and empty spacetime is globally hyperbolic and every Cauchy surface is homeomorphic to $I\!R^3$. Either component $\mathcal{J}^\pm$ of ∂M can be diffeomorphically mapped onto $S^2 \times I\!R$ in such a way that each generator of $\mathcal{J}^\pm$ is mapped onto an $I\!R$-line. Here S^2 denotes the 2-dimensional sphere.

3 The lens map for light sources at infinity

An interesting feature of asymptotically simple and empty spacetimes is in the fact that they allow to discuss gravitational lensing situations with light sources at infinity. To make this precise we consider, in an asymptotically simple and empty spacetime, a sequence of timelike C^∞ curves $\gamma_n : I \longrightarrow M$ that approach, for $n \to \infty$, a curve $\gamma : I \longrightarrow \mathcal{J}^-$. Here I denotes a real interval. We want to assume that γ is an immersed curve of class C^1 at least, and that the limit is in the C^1 sense, i.e., that not only $\lim_{n\to\infty} \gamma_n(s) = \gamma(s)$ in $\tilde{M}$ but also $\lim_{n\to\infty} \gamma_n'(s) = \gamma'(s)$ in $T\tilde{M}$. Since $\gamma_n'(s)$ is g-timelike and, thus, $\tilde{g}$-timelike, $\gamma'(s)$ is either $\tilde{g}$-timelike or $\tilde{g}$-lightlike. The first case is impossible, since $\mathcal{J}^-$ is a $\tilde{g}$-lightlike hypersurface, and the second case is possible only if $\gamma'(s)$ is tangent to a generator of $\mathcal{J}^-$. We are thus led to the following conclusion. In an asymptotically simple and empty spacetime, the worldline of a light source at infinity is to be identified with (a section of) a generator of $\mathcal{J}^-$.

Please note that this does *not* mean that light sources at infinity move at the speed of light. The (physical) metric g is not defined on ∂M, and the causal character of curves in ∂M with respect to the (unphysical) metric $\tilde{g}$ has no physical interpretation.

Henceforth we restrict to light sources at infinity with inextendible worldlines, i.e., to (maximal) generators of $\mathcal{J}^-$. From Theorem 2.2. we know that the set of generators of $\mathcal{J}^-$ is a manifold diffeomorphic to S^2. Hence, the set of all light sources at infinity is in one-to-one correspondence with the points of S^2. On the other hand we can consider for any $p \in M$ the set of all one-dimensional g-lightlike subspaces of T_pM. This, again, gives a manifold diffeomorphic to S^2 which may be called the *sky* at p since each point of this manifold corresponds to a light ray arriving at p. In this way we get, for each $p \in M$, a map

$$f_p : S^2 \longrightarrow S^2 \tag{1}$$

by assigning to each point x of the sky at p a light source $f_p(x)$ at infinity by extending the lightlike geodesic tangent to x until it reaches $\mathcal{J}^- \simeq S^2 \times \mathbb{R}$ and projecting onto the first factor afterwards. Henceforth we refer to this map f_p as to the *lens map* at p for light sources at infinity. For each light source at infinity, represented by a point $y \in S^2$, the set $f_p^{-1}(y)$ gives all points of the sky at p where this light source is seen. Gravitational lensing situations are characterized by the fact that $f_p^{-1}(y)$ consists of two or more points.

Please recall that $y \in S^2$ is called a *regular value* of f_p if for all $x \in S^2$ with $f_p(x) = y$ the differential $T_xf_p : T_xS^2 \longrightarrow T_yS^2$ has maximal rank, i.e., is surjective. It is easy to check that y is a regular value of the lens map f_p if and only if the generator represented by y does not intersect the caustic of the past light cone of p. (One may use the metric $\tilde{g}$ to extend the "physical" light cone to $\mathcal{J}^-$.) Owing to the well-known Theorem of Sard (see, e.g., Guillemin and Pollack [7], p. 39) almost all points $y \in S^2$ are regular values of the lens map. By compactness of S^2, $f_p^{-1}(y)$ is finite for any regular value y. Hence, the observer at p sees finitely many images of each light source at infinity that does not pass through the caustic of the past light cone of p. In the next section we shall indicate that this finite number of images is always odd, and that the same is true for a light source moving inside M provided that its worldline is inextendible and stays away from $\mathcal{J}^-$.

4 The odd number theorem

For the lens map (1), we choose a regular value y of f_p and define

$$\deg(f_p) = \sum_{x \in f_p^{-1}(y)} \operatorname{sgn}(x) \tag{2}$$

where $\operatorname{sgn}(x)$ is, by definition, equal to $+1$ if the tangent map $T_x f_p$ preserves orientation and equal to -1 if $T_x f_p$ reverses orientation. $\deg(f_p)$ is called the *degree* of f_p, cf., e.g., Guillemin and Pollack [7] or Dold [8]. Using the fact that S^2 is compact without boundary, it is a standard result that $\deg(f_p)$ is, indeed, independent of y. Moreover, the degree is a homotopic invariant in the sense that a second (C^1) map $f : S^2 \longrightarrow S^2$ has the same degree as f_p if f_p can be continuously deformed into f. We find the following result.

Theorem 4.1. For any point p in an asymptotically simple and empty spacetime, the lens map (1) has degree one, $\deg(f_p) = 1$.

The proof of Theorem 4.1. is based on a continuous deformation of the lens map into the identity using the exponential map of the metric $\tilde{g}$. A similar result was put forward by Kozameh, Lamberti and Reula [9], Lemma 1.

For a regular value $y \in S^2$ of the lens map f_p, we denote by $n_\pm$ the number of points $x \in f_p^{-1}(y)$ such that $\operatorname{sgn}(x) = \pm 1$. Then Theorem 4.1. implies $n_+ - n_- = 1$, so the number of points in $f_p^{-1}(y)$ is odd, $n_+ + n_- = 2n_- + 1$.

In addition to this odd number theorem for light sources at infinity, one may establish a similar result for light sources inside M.

Theorem 4.2. Fix a point p in an asymptotically simple and empty spacetime (M, g). Let I be an open interval and $\gamma : I \longrightarrow M$ a timelike embedded C^∞ curve such that the image of γ is topologically closed in M and γ has no endpoint on $\mathcal{J}^-$. (This is a way of saying that γ is inextendible in M and, in the past direction, does not go out to infinity approaching the velocity of light.) If γ does not pass through the caustic of the past light cone of p, then the number of past-pointing lightlike geodesics from p to γ is finite and odd.

The proof of Theorem 4.2. is based on Theorem 4.1. and uses the local degree for maps between manifolds with boundaries which is discussed, e.g., by Dold [8]. Detailed proofs of Theorem 4.1. and Theorem 4.2. will be published elsewhere.

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