Why do we need nonautonomous models and methods?

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1. Nonautonomicity and real life

A system is said nonautonomous when its **evolution law** is **time-varying**.

Such time-variable dynamics can be the **result of a time-evolving environment** which unidirectionally influences the system.

For example: the influence of **circadian and seasonal rhythms** on the human body.

To date, nonautonomous systems are **mostly treated as autonomous or stochastic**, because of their seemingly complicated dynamics.

2. Mathematical description

Nonautonomous systems either have an explicit time-dependence

$$\dot{x}_1(t) = f_1(x_1, \dots, x_n, t)$$
$$\vdots$$
$$\dot{x}_n(t) = f_n(x_1, \dots, x_n, t)$$

or can be written in the skew-product formalism [2]

$$\begin{cases} \dot{\mathbf{p}} = \mathbf{f}(\mathbf{p}) \\ \dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{p}(\mathbf{t})) \end{cases}$$

where the $\mathbf{p}$-system is a time-varying unidirectional influence on $\mathbf{x}$.

3. Example: forced Poincaré oscillator

The system can be written as

$$\dot{x} = -qx - \omega y$$
$$\dot{y} = \omega x - qy + \gamma f_\gamma(t)$$
$$q = \alpha(\sqrt{x^2 + y^2} - a)$$

with the non-periodic forcing $f_\gamma(t) = \sin(2\pi t) + \sin(4t)$ [1].

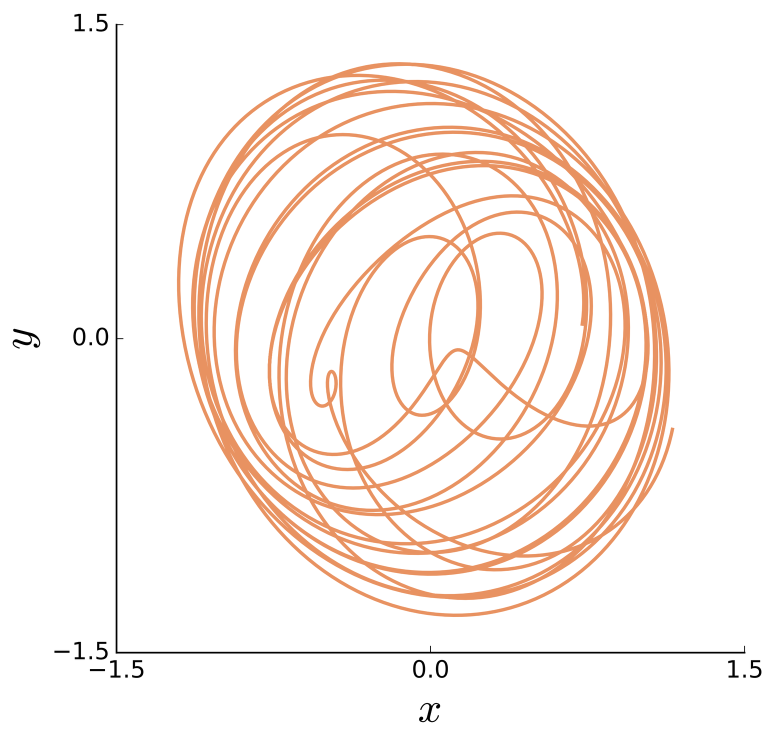

Fig. 1: Forced Poincaré oscillator. One trajectory in state space: no clear attractor.

Can one not add one variable and make it autonomous?

4. Classical view: yes one can

One can add an extra dimension accounting for time [3], $x_{n+1} \equiv t$, and rewrite

$$\dot{x}_1(t) = f_1(x_1, \dots, x_n, x_{n+1})$$
$$\vdots$$
$$\dot{x}_n(t) = f_n(x_1, \dots, x_n, x_{n+1})$$
$$\dot{x}_{n+1}(t) = 1$$

This extended system evolves in the extended state space.

But is this of any use? What can one do next?

5. Extended state space: is there an attractor?

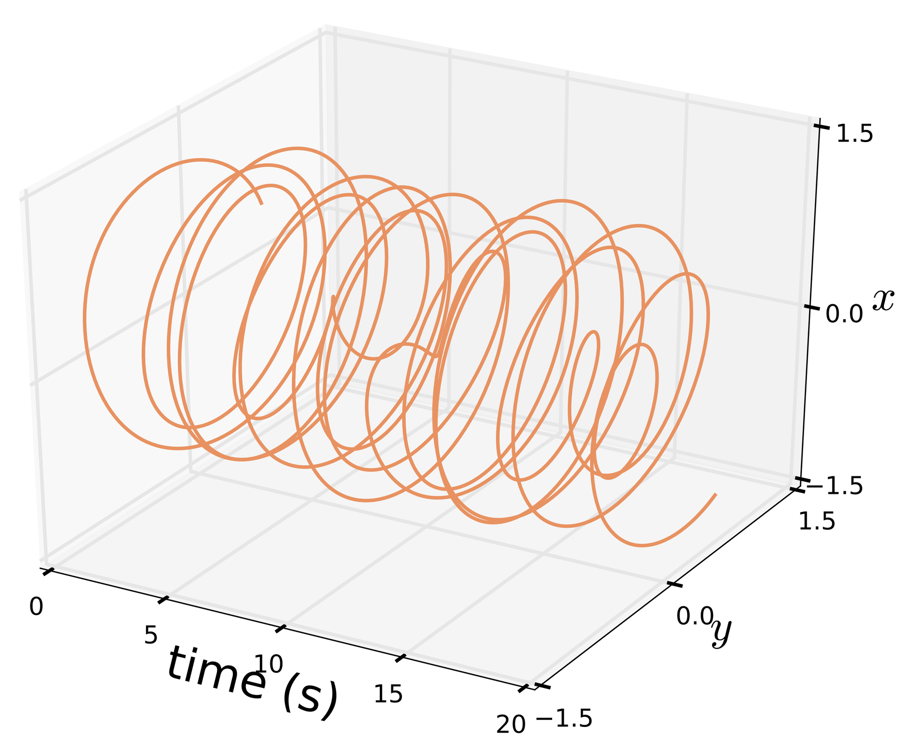

Fig. 2: Forced Poincaré oscillator. Extended state space.

6. Problem: autonomous methods cannot be applied

Problems:

- the extra variable is **unbounded**
- hence **no fixed point exists**
- a trajectory will **never return to a same region** of the extended phase space

Classical autonomous methods cannot be applied. For example:

- time-delay embedding only produces bounded trajectories in reconstructed state space
- cannot compute the dimension of the attractor

Hence, **time cannot be forgotten**, and new definitions and methods are needed.

For example, the concepts of **pullback attraction** [2] and **time-varying point attractor** [4] only exist in a nonautonomous framework.

7. Yes, there is a one-dimensional attractor

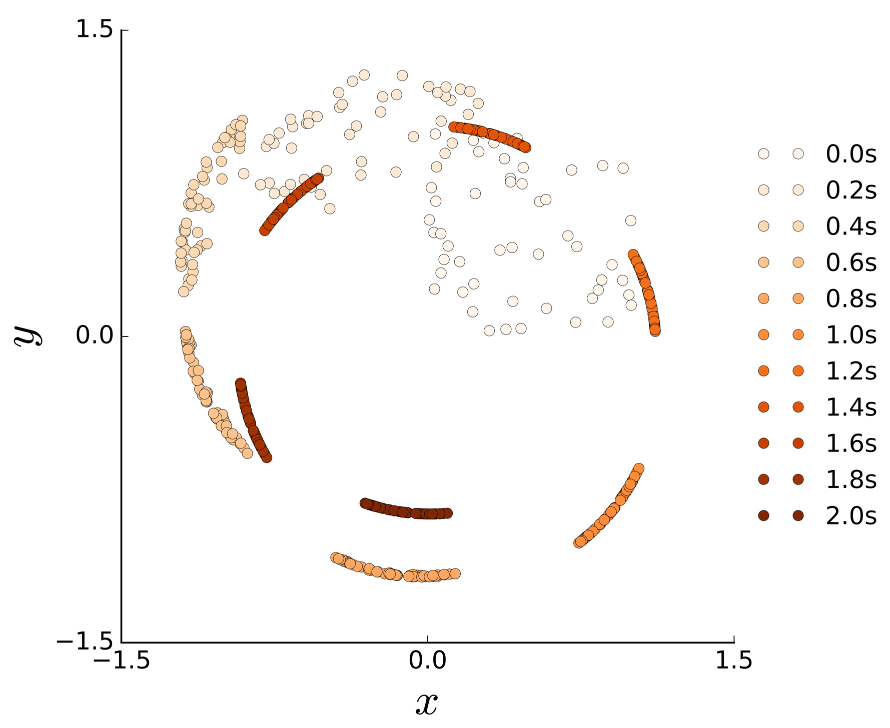

Fig. 3: Forced Poincaré oscillator. 40 trajectories in state space for fixed times.

What next?

Describe nonautonomous systems with truly nonautonomous models and methods, which will allow one to have more insight about the inner functioning of the system.
See for example [4] and references therein.

Interested?

[1] Philip T. Clemson and Aneta Stefanovska. "Discerning Non-Autonomous Dynamics". In: *Phys. Rep.* 542.4 (2014), pp. 297–368.

[2] Peter E. Kloeden and Martin Rasmussen. *Nonautonomous Dynamical Systems*. Providence: Amer. Math. Soc., 2011.

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[4] Yevhen F. Suprunenko, Philip T. Clemson, and Aneta Stefanovska. "Chronotaxic Systems: A new class of Self-Sustained Nonautonomous Oscillators". In: *Phys. Rev. Lett.* 111.2 (2013), p. 024101.