Is it always easier to capture time-varying dynamics which resist the influence of external perturbations?

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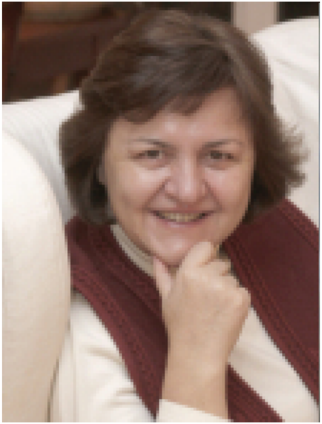
Introduction

My work in Lancaster group together with:



Prof. Aneta Stefanovska

Dr Phil Clemson

Dr Tomislav Stankovski

Dr Gemma Lancaster

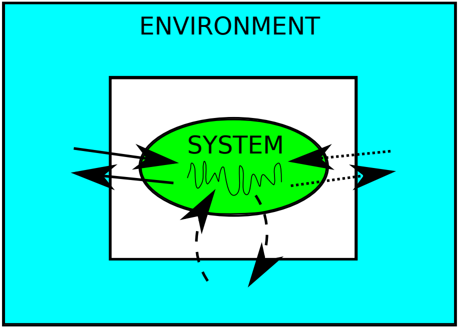
Outline

1. Time-varying dynamics?
2. Ability to resist external perturbations?
3. Nonautonomous dynamical systems
4. Chronotaxic systems
5. Cardio-respiratory system

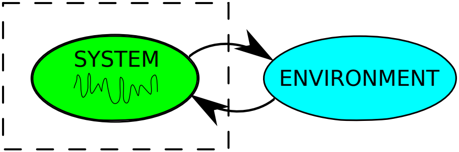
Time-varying dynamics?

Nonautonomous (time-varying) dynamics in living (open) systems:

$$\dot{x} = f(x, t)$$

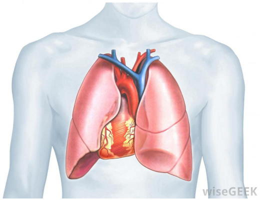


Time-varying dynamics?



Examples, just to name a few:

- in cardio-respiratory system



http://goo.gl/QXjI0v

- in cells (metabolic oscillations)

http://goo.gl/SH1tmq

Ability to resist external perturbations?

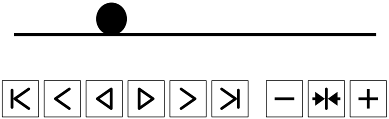
Living systems:

- STRUCTURE: cells, tissues, blood vessels, organs, etc.
  - often, well defined objects
  - maintain their structure over long time
  - certain stability
- FUNCTION: heart beat, respiration, etc.
  - maintenance of stable internal environment
  - certain stability

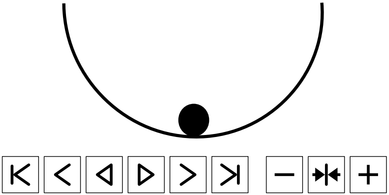
Ability to resist external perturbations?

Simple mechanical analogies:

- NOT STABLE function:



- STABLE function



Oscillations with phase ϕ

$$\dot{\phi} = \omega$$

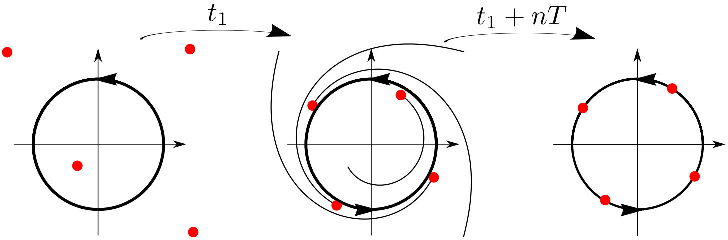
$$\dot{\phi} = \omega + \delta\omega$$

$$\begin{aligned} \dot{\phi} &= \omega - \sin(\phi - \theta) \\ \dot{\theta} &= \omega \end{aligned}$$

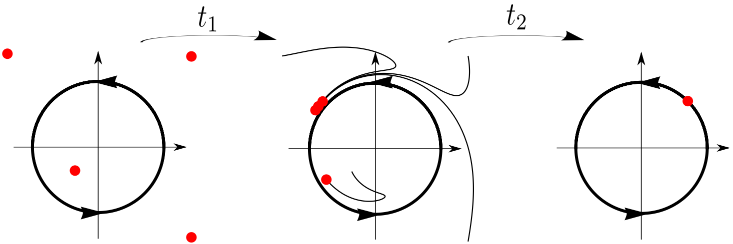
$$\psi = \phi - \theta, \quad \dot{\psi} = -\sin\psi$$

Ability to resist external perturbations?

- NOT STABLE phase of oscillations



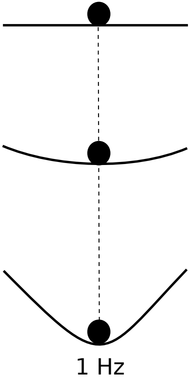
- STABLE phase of oscillations



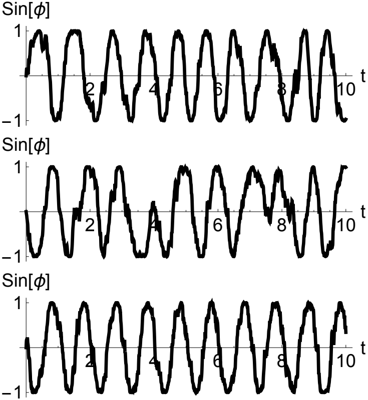
Ability to resist external perturbations?

Example: artificial heart with frequency $f = 1\mathrm{Hz}$, $\dot{\phi} = g(f, \phi, t) + \xi(t)$

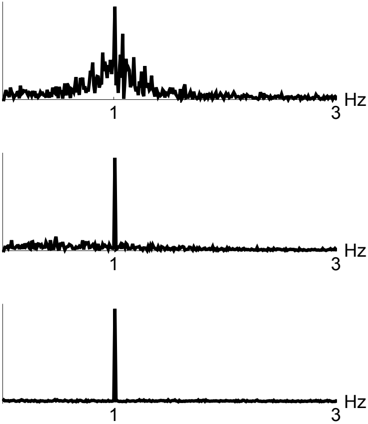
Ability to resist:



Dynamics:



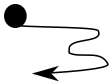
Fourier transform:



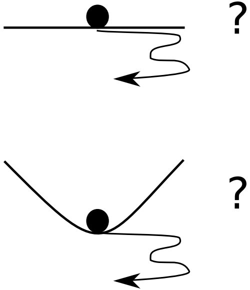
Ability to resist external perturbations?

When frequency is time-dependent, when time-varying dynamics is observed...

Observed dynamics:



Does it resist perturbations?



How to quantify the ability to resist perturbations in time-varying dynamics? ... using nonautonomous systems!

Nonautonomous dynamical systems

Nonautonomous systems –

- Skew-product flow from unidirectionally coupled differential equations,

$$\dot{\mathbf{p}} = \mathbf{f}(\mathbf{p})$$
$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, \mathbf{p})$$

- $\mathbf{p} \in R^n$, $\mathbf{x} \in R^m$,
- $\mathbf{p}$ drives the nonautonomous system $\mathbf{x}$
- $\mathbf{x}$ is dependent on t, t_0 and $\mathbf{x}_0$.

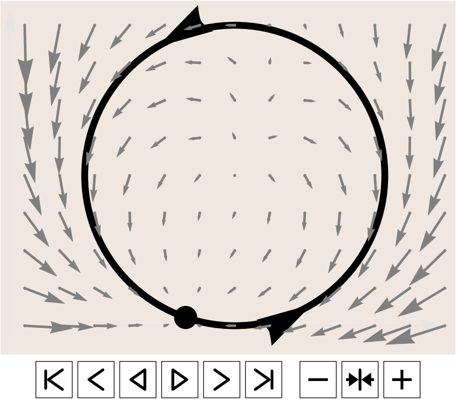
Thus a system is described by:

$$\frac{d\mathbf{x}}{dt} = \tilde{\mathbf{g}}(\mathbf{x}, t)$$

Kloeden and Rasmussen, *Nonautonomous Dynamical Systems* (2011)

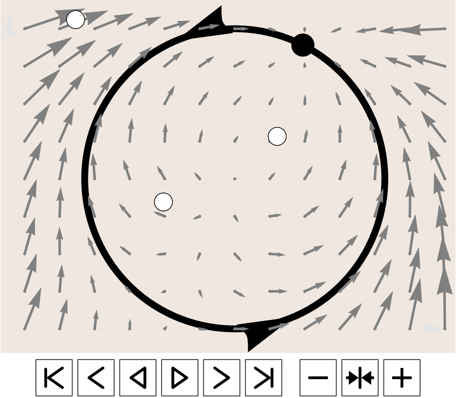
Nonautonomous dynamical systems

In polar coordinates (r, φ)
System $\mathbf{p}$: $\dot{r}_{\mathbf{p}} = 0$; $\dot{\varphi}_{\mathbf{p}}(t) = \omega_{\mathbf{p}}(t)$;
System $\mathbf{x}$: $\dot{r}_{\mathbf{x}} = \varepsilon_r r_{\mathbf{x}}(r_{\mathbf{p}} - r_{\mathbf{x}})$; $\dot{\varphi}_{\mathbf{x}}(t) = -\varepsilon \sin(\varphi_{\mathbf{x}} - \varphi_{\mathbf{p}})$;
Black disk = $(r_{\mathbf{p}}, \varphi_{\mathbf{p}}(t))$. Gray arrows = velocities $\dot{\mathbf{x}}$; $\varepsilon > \omega_{\mathbf{p}}(t) > 0$.

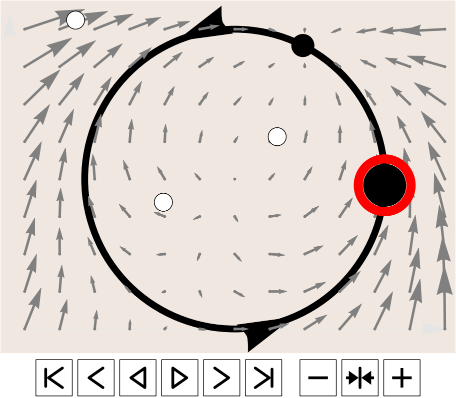


Nonautonomous dynamical systems

Trajectories of $\mathbf{x}$:



Time-dependent point attractor exists:



Time-dependent point attractor

Time-dependent point attractor (driven steady state) $\mathbf{x}^A(t)$.

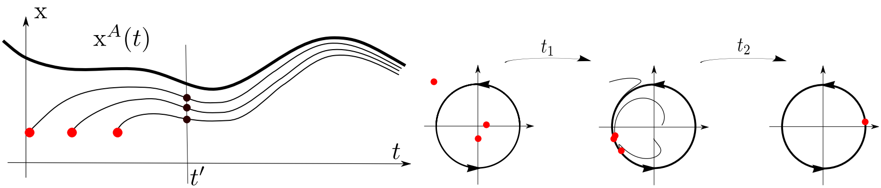
$\mathbf{x}^A(t)$ satisfies mathematical conditions of pullback attraction (1), forward attraction (2) and invariance (3):

$$\lim_{t_0 \to -\infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{x}^A(t) \quad (1)$$

$$\lim_{t \to \infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{x}^A(t) \quad (2)$$

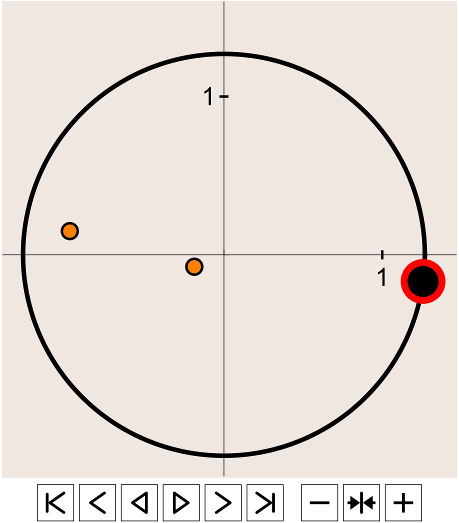
$$\mathbf{x}(t, t_0, \mathbf{x}^A(t_0)) = \mathbf{x}^A(t) \quad (3)$$

Kloeden and Rasmussen, *Nonautonomous Dynamical Systems* (2011)



Nonautonomous dynamical systems

Time-dependent point attractor in phase and amplitude dynamics:



Chronotaxic systems

We therefore define a **new class** of nonautonomous oscillators: **chronotaxic systems** (from *chronos* – time and *taxis* – order).

Suprunenko, Clemson and Stefanovska, *PRL* (2013); *PRE* (2014)

Chronotaxic Systems: Defining concepts

1) Nonautonomous systems.

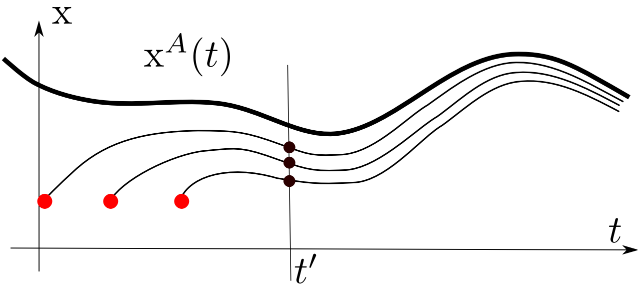
$$\frac{d\mathbf{x}}{dt} = \mathbf{g}(\mathbf{x}, t); \qquad (4)$$

2) Time-dependent point attractor (driven steady state) $\mathbf{x}^A(t)$.

$$\lim_{t_0 \to -\infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{x}^A(t); \qquad (5)$$

$$\lim_{t \to \infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{x}^A(t); \qquad (6)$$

$$\mathbf{x}(t, t_0, \mathbf{x}^A(t_0)) = \mathbf{x}^A(t). \qquad (7)$$



P. E. Kloeden and M. Rasmussen, Nonautonomous Dynamical Systems (American Mathematical Soc., 2011)

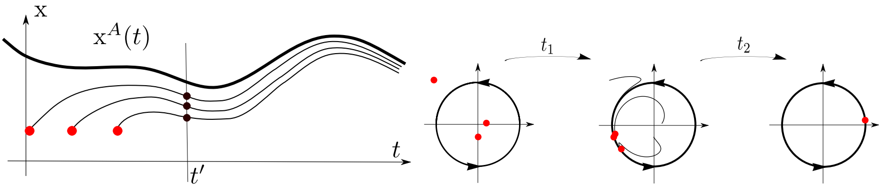
Chronotaxic systems: Definition

Chronotaxic systems are nonautonomous oscillatory systems $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, t)$ with solutions $\mathbf{x}(t, t_0, \mathbf{x}_0)$ and with a **time-dependent point attractor (driven steady state) $\mathbf{x}^A(t)$** which satisfies mathematical conditions of pullback attraction, forward attraction and invariance:

$$\lim_{t_0 \to -\infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{x}^A(t)$$
$$\lim_{t \to \infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{x}^A(t)$$
$$\mathbf{x}(t, t_0, \mathbf{x}^A(t_0)) = \mathbf{x}^A(t)$$

Additional requirement: attraction at all times, deviations $\delta\mathbf{x}^A(t) = \mathbf{x}(t) - \mathbf{x}^A(t)$ from point attractor $\mathbf{x}^A(t)$ in unperturbed system can only decay

$$\frac{d}{dt}|\delta\mathbf{x}^A|^2 = 2\delta\mathbf{x}^{A\,T} J(\mathbf{x}^A, t)\delta\mathbf{x} < 0 \quad \text{and} \quad J(\mathbf{x}, t) = \frac{1}{2}\left(\partial\mathbf{g}/\partial\mathbf{x} + \partial\mathbf{g}/\partial\mathbf{x}^T\right)$$



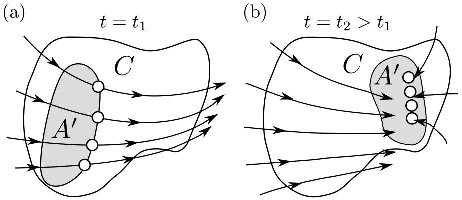
Chronotaxic systems: Alternative definition

- Alternative formulation is based on the contraction theory (*W. Lohmiller and J.-J. Slotine (1998)*). The explicit knowledge of $\mathbf{x}^A(t)$ is not required.

A chronotaxic system is an oscillatory nonautonomous dynamical system $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, t)$ which has a contraction region C in the phase space, C is determined by

$$\forall \mathbf{x} \in C \;\; \exists \beta > 0, \forall t \;\; : \qquad \frac{1}{2}\left(\frac{\partial \mathbf{g}(\mathbf{x}, t)}{\partial \mathbf{x}} + \frac{\partial \mathbf{g}(\mathbf{x}, t)}{\partial \mathbf{x}}^T\right) \leq -\beta I < 0,$$

C contains a finite non-zero area $A' \subset C$ such that states of a system inside A' do not leave A', i.e. $\forall t_0 < t, \;\; \forall \mathbf{x}_0 \in A'(t_0), \mathbf{x}(t, t_0, \mathbf{x}_0) \in A'(t)$.



Y. S., Stefanovska A., PRE (2014)

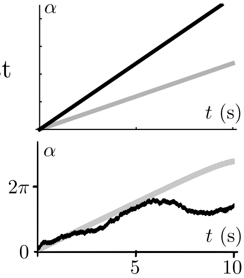
Chronotaxic systems: Phase oscillators

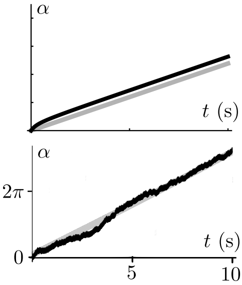
Phase α of the phase oscillator which is

conventional	chronotaxic
does not resist external perturbations ξ, as shown in this column:	is able to resist external perturbations ξ, as shown in this column:
$\dot{\alpha} = \omega_0 + \xi$	$\dot{\alpha} = g_\alpha(e^{i\alpha_x}, t) + \xi$ $\frac{\partial}{\partial\alpha} g_\alpha(e^{i\alpha_x}, t)\big\|_{\alpha=\alpha^A(t)} < 0.$

$\xi = \mathrm{Const}$

$\xi = \xi(t)$





Y.F.S., Clemson P.T., Stefanovska A., PRE (2014)

Chronotaxic systems: Poincare oscillator

Nonautonomous (driven) Poincare oscillator:

$$\dot{x} = \varepsilon_\Gamma \left(r_p - \sqrt{x^2 + y^2} \right) x - \omega_0 y - \varepsilon_A(t)\left(x - r_p \cos\varphi_p(t)\right) + \xi_1(t),$$
$$\dot{y} = \varepsilon_\Gamma \left(r_p - \sqrt{x^2 + y^2} \right) y + \omega_0 x - \varepsilon_A(t)\left(y - r_p \sin\varphi_p(t)\right) + \xi_2(t).$$

- nonautonomous system:
  p is defined by polar coordinates $r_p, \varphi_p(t)$;
  x is defined by x, y;
- $\varepsilon_A(t)$ determines coupling;
- perturbations are defined by $\xi_1(t), \xi_2(t)$;
- phase and amplitude dynamics are coupled.

Y.F.S., Stefanovska A., PRE (2014)

Nonautonomous (driven) Poincare oscillator: $\dot{\varphi}_p(t) = \omega_p(t) = \mathrm{const};$

$$\begin{aligned}
\dot{x} &= \varepsilon_r \left(r_p - \sqrt{x^2 + y^2} \right) x - \omega_0 y - \varepsilon_A(t) \left(x - r_p \cos \varphi_p(t) \right) + \xi_1(t), \\
\dot{y} &= \varepsilon_r \left(r_p - \sqrt{x^2 + y^2} \right) y + \omega_0 x - \varepsilon_A(t) \left(y - r_p \sin \varphi_p(t) \right) + \xi_2(t).
\end{aligned}$$

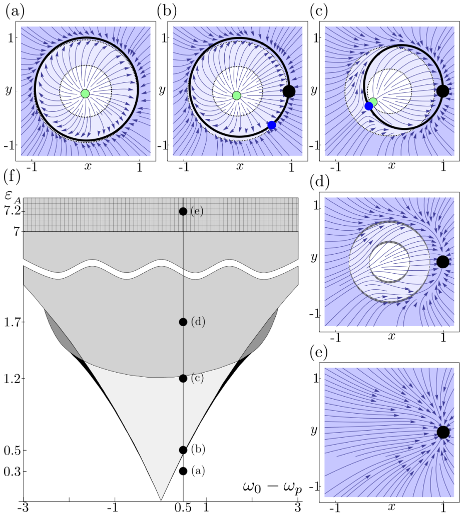


Figure shows:

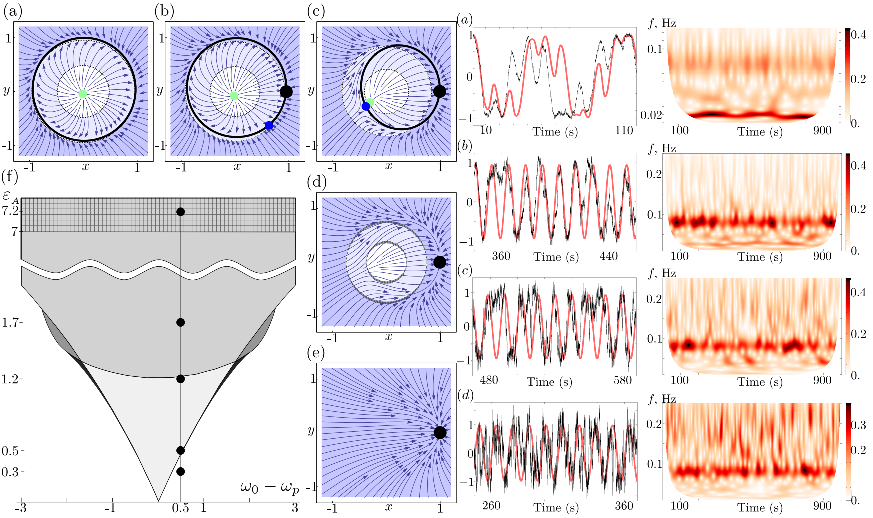
- snapshots of phase portrait of this system when it is:

  (a) non-chronotaxic;

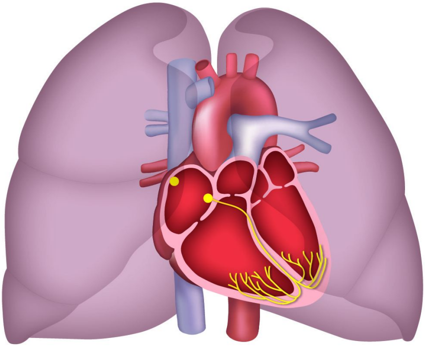
  (b),(c),(d),(e) - chronotaxic (time-dependent point attractor is shown as a black disk, the green point is an unstable node, the blue point is a saddle point);
- (f) regions of different chronotaxic dynamics.

Y.F.S., Stefanovska A., PRE (2014)

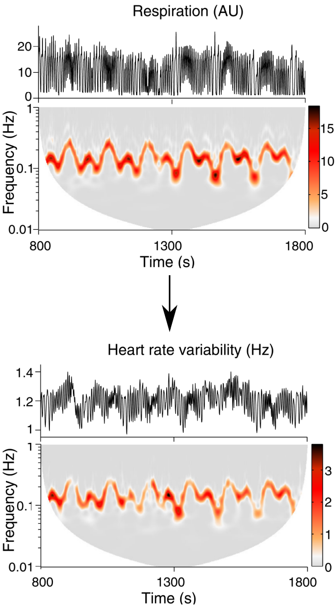
- Parameters: white Gaussian noise $\langle \xi_j(t) \rangle = 0$, $\langle \xi_i(t)\xi_j(t') \rangle = \sigma^2 \delta_{ij}\delta(t - t')$, $\omega_p/(2\pi) \approx 0.08\mathrm{Hz}$, $\varepsilon_\Gamma = 7$, $r_p = 1$; (a) $\varepsilon_A = 0.3$, $\sigma = 0.1$; (b) $\varepsilon_A = 0.47$, $\sigma = 0.3$ (c) $\varepsilon_A = 0.47$, $\sigma = 0.6$ (d) $\varepsilon_A = 0.9$, $\sigma = 1.2$.
- In chronotaxic systems (b),(c),(d) the frequency $\omega_p/(2\pi)$ determines a frequency of 0.08Hz which resists external perturbations, as shown in wavelet transforms;



Cardio-respiratory system



Cardio-respiratory system



*Y.F.S., Clemson P.T., Stefanovska A., PRL **111**, 024101 (2013)*

Clemson P.T., Y.F.S., Stefanovska A., PRE (2014).

(1) The paced respiration drives HRV;

(2) We are looking for a new feature - a stability of a time-varying dynamics;

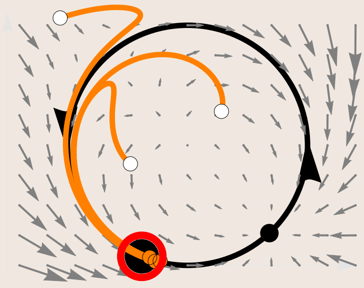
(3) Stability in time-varying HRV is found using a single time-series, and checked using both time-series;

(4) Alterations in chronotaxicity may help to distinguish normal and altered states.

Summary

Chronotaxicity as an ability to resist external perturbations:

- helps to capture time-variability;
- is a quantifiable characteristics;
- may help to identify subsystems which maintain their function, and to clarify functional organization within a living system.



Thank you for your attention!