9th meeting of

**E**uropean
**S**tudy
**G**roup on
**C**ardiovascular
**O**scillations


# Approximate and Identical Synchronization in Coupled Systems

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April 13, 2016

# Coupled systems

$$\text{ODE}: \quad \frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t), \;\; \mathbf{x} = (x_1, \ldots, x_K)$$

$$\text{ODE}: \quad \frac{d\mathbf{y}}{dt} = \tilde{\mathbf{F}}(\mathbf{y}, t), \;\; \mathbf{y} = (y_1, \ldots, y_K)$$

Coupled systems

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) + \mathbf{G}(\mathbf{x}, \mathbf{y}, t) \\ \dfrac{d\mathbf{y}}{dt} = \tilde{\mathbf{F}}(\mathbf{y}, t) + \tilde{\mathbf{G}}(\mathbf{y}, \mathbf{x}, t) \end{cases}$$

$$e.g. \;\; \mathbf{G}(\mathbf{x}, \mathbf{y}, t) = W \cdot (\mathbf{y} - \mathbf{x})$$

$$\tilde{\mathbf{G}}(\mathbf{y}, \mathbf{x}, t) = \tilde{W} \cdot (\mathbf{x} - \mathbf{y})$$

linear coupling

# Delay equations and coupled-cell systems

- Single cell

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}_t, t), \qquad \mathbf{x}(t) \in \mathbb{R}^K$$

$$\mathbf{x}_t(\theta) = \mathbf{x}(t+\theta), \ \theta \in [-\tau_M, 0]$$

internal delays, time delay in intracellular process

- Coupled cells (cell-to-cell)

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}_t, t) + \mathbf{G}(\mathbf{x}_t, \mathbf{y}_t, t) \\ \frac{d\mathbf{y}}{dt} = \tilde{\mathbf{F}}(\mathbf{y}_t, t) + \tilde{\mathbf{G}}(\mathbf{y}_t, \mathbf{x}_t, t) \end{cases}$$

transmission delays, time delay in intercellular process

# Linearly coupled systems with delays

*e.g.* linear coupling with commnication delay (discrete-type)

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x},t) + k \bullet [\mathbf{y}(t-\tau) - \mathbf{x}(t)] \\ \dfrac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y},t) + k \bullet [\mathbf{x}(t-\tau) - \mathbf{y}(t)] \end{cases}$$

# Coupled cells and synchronization

When $\mathbf{F} = \tilde{\mathbf{F}}, \mathbf{G} = \tilde{\mathbf{G}}$: identical subsystems, symmetric coupling

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}_t, t) + \mathbf{G}(\mathbf{x}_t, \mathbf{y}_t, t) \\ \dfrac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y}_t, t) + \mathbf{G}(\mathbf{y}_t, \mathbf{x}_t, t) \end{cases}$$

Coupled system attains globally identical synchronization if

$$\mathrm{x}_i(t) - y_i(t) \to 0, \text{ as } t \to \infty, \; i = 1, \; ..., \; K$$

for all solutions $(\mathbf{x}(t), \mathbf{y}(t))$, where

$$\mathbf{x}(t) = (x_1(t), \ldots, x_K(t)),$$
$$\mathbf{y}(t) = (y_1(t), \ldots, y_K(t)),$$

# Synchronous set

In this case, synchronous set $S := \{(\mathbf{x},\mathbf{y}) : \mathbf{x} = \mathbf{y}\}$

is positively invariant

$\mathbf{x}(t)$ is a solution of $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}_t, t) + \mathbf{G}(\mathbf{x}_t, \mathbf{x}_t, t)$

$\Rightarrow (\mathbf{x}(t), \mathbf{x}(t))$ is a solution of $\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}_t, t) + \mathbf{G}(\mathbf{x}_t, \mathbf{y}_t, t) \\ \dfrac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y}_t, t) + \mathbf{G}(\mathbf{y}_t, \mathbf{x}_t, t) \end{cases}$

# Coupled cells and synchronization

When $\mathbf{F} \neq \tilde{\mathbf{F}}, \mathbf{G} \neq \tilde{\mathbf{G}},$

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}_t, t) + \mathbf{G}(\mathbf{x}_t, \mathbf{y}_t, t) \\ \dfrac{d\mathbf{y}}{dt} = \tilde{\mathbf{F}}(\mathbf{y}_t, t) + \tilde{\mathbf{G}}(\mathbf{y}_t, \mathbf{x}_t, t) \end{cases}$$

the synchronous set $S := \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} = \mathbf{y}\}$ is not positively invariant in general

Coupled system attains (globally) approximate synchronization if

$$|x_i(t) - y_i(t)| < \varepsilon, \text{ as } t \to \infty, \; i = 1, \; \ldots, \; K$$

for all solutions $(\mathbf{x}(t), \mathbf{y}(t))$, where

$$\mathbf{x}(t) = (x_1(t), \ldots, x_K(t)), \; \mathbf{y}(t) = (y_1(t), \ldots, y_K(t)),$$

This $\varepsilon$, synchronization error, would be meaningful if it depends and decreases with decreasing $|\delta F|$ and $|\delta G|$

where $|\delta F| = | F - \tilde{F} |$

$|\delta G| = | G - \tilde{G} |$

# Coupled network systems

Coupled $N$ subsystems

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i^t, t) + \sum_{j\in\mathbf{N}} w_{ij}(t)\ \mathbf{G}_{ij}(\mathbf{x}_i^t, \mathbf{y}_j^t, t), \quad i \in \mathbf{N} := \{1, \ldots, N\}$$

$$\mathbf{x}_i^t(\theta) = \mathbf{x}_i(t+\theta),$$

$W(t) = [w_{ij}(t)]$ : coupling matrix

$\kappa_i(t) = \sum_{j\in\mathbf{N}} w_{ij}(t)$: $i$th row sum of coupling matrix

The coupled system is called homogeneous if $\mathbf{F}_i = \mathbf{F}_j$, for all $i$, $j$

The synchronous set $S := \left\{(\mathbf{x}_1, \ldots, \mathbf{x}_N) : \mathbf{x}_i = \mathbf{x}_j, \forall i, j = 1, \ldots, N\right\}$

is positively invariant

if the coupled system is homogeneous, has identical row sums and delays

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i^t, t) + \sum_{j \in \mathbf{N}} w_{ij}(t)\, \mathbf{G}_{ij}(\mathbf{x}_i^t, \mathbf{y}_j^t, t), \quad i \in \mathbf{N} := \{1, \ldots, N\}$$

- linear coupling: $\mathbf{G}_{ij}$ is linear
- diffusive coupling: $\kappa_i \equiv 0$, for all $i$

$\kappa_i(t) = \sum_{j \in \mathbf{N}} w_{ij}(t)$: $i$th row sum of coupling matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

# Heterogeneous systems

- Due to various possible deviations, natural imperfection, and wider consideration such as real-world complex networks, heterogeneous networks which consist of non-identical nodes are more practical.

- In neural connections, there are gap-junctional (diffusive) and chemical synaptic coupling (nonlinear, not necessarily diffusive)

- In gene regulation, the coupling (connection, reception, binding)  among cells is not linear, nor diffusive.

# Notions of synchronization

- Identical synchronization for homogeneous coupled systems
- Approximate synchronization for heterogeneous coupled systems
- Asymptotic synchronization for heterogeneous coupled systems

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i^t, t) + c \cdot \sum_{j \in \mathbf{N}} w_{ij}(t)\, \mathbf{G}_{ij}(\mathbf{x}_i^t, \mathbf{y}_j^t, t), \quad i \in \mathbf{N} := \{1, \ldots, N\}$$

$$|x_i(t) - y_i(t)| < \varepsilon, \text{ as } t \to \infty, \; i = 1, \ldots, n$$

$$\varepsilon \to 0, \text{ as } c \to \infty$$

Does asymptotic synchronization always hold when approximate synchronization holds ?

Jack Hale
[Diffusive coupling, dissipation, and synchronization, J. Dynamics and Differential Equations, 1997]

Via theory of invariant manifold and perturbation property

e.g. coupled Lorenz systems

# Research on synchronization

- Methodologies for concluding global synchronization largely involve the notion of Lyapunov functions.
- Phase synchronization, lag synchronization, partial synchronization, generalized synchronization, and almost synchronization
- Identical synchronization of homogeneous networks is the simplest form of synchronization, and has been intensively studied.
- Most of the works require diffusive coupling, and/or linear coupling, due to mathematical treatment.

# Approximate synchronization

We consider approximate synchronization for heterogeneous networks which consist of not necessarily identical subsystems

Synchronous set $S := \left\{ (\mathbf{x}_1, \ldots, \mathbf{x}_N) : \mathbf{x}_i = \mathbf{x}_j, \forall i, j = 1, \ldots, N \right\}$

is not positively invariant in general.

Some methods for studying identical synchronization are not applicable.

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i^t, t) + \sum_{j \in \mathbf{N}} w_{ij}(t)\, \mathbf{G}_{ij}(\mathbf{x}_i^t, \mathbf{y}_j^t, t), \quad i \in \mathbf{N} := \{1, \ldots, N\}$$

Let $z_{i,k} = x_{i,k} - x_{i+1,k}$

Via sequential contracting, estimate $|z_{i,k}(t)|$ successively, and iteratively, and converted the lengths of intervals where $z_{i,k}(t)$ lie to the Gauss-Seidel iteration for a system of linear algebraic equations.

Condition of convergence for this Gauss-Seidel iteration then leads to synchronization criterion.

We established a theorem which gives a delay-dependent criterion and a delay-independent criterion for approximate synchronization of the general coupled network systems.

# Conclusion

- We have developed a mathematical framework to investigate approximate synchronization for the general coupled systems, and network systems.

- Approximate synchronization yields asymptotic synchronization in some cases, but not all.

- Linearization over-manipulates the nonlinear terms.

- Sequential contracting: start with a preliminary estimate on the absorbing set, use mean value theorem, some estimates, structure about dissipation in the equations, iterative argument.

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Shih & Tseng, “A general approach to synchronization of coupled cells” , 2013, *SIAM J. Applied Dynamical Systems*

Shih & Tseng, “From approximate synchronization to identical synchronization in coupled systems”, *preprint*

Thank you for your attention!