Solvable model for a network of spiking neurons with delays

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Abstract

Which are the underlying mechanism of neuronal oscillations?

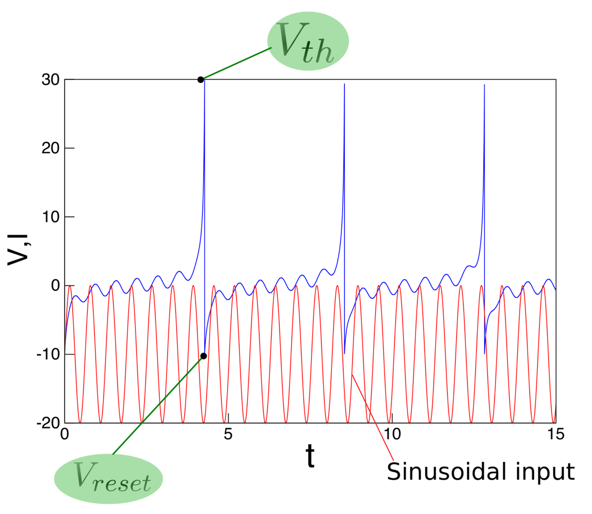
Collective behaviours that arise in large networks of interacting neurons are known to play a crucial role in processing and coding of information in the brain. Here we analyze a network of Quadratic Integrate-and-Fire (QIF) neurons with delayed synaptic interaction. With a dimensionality reduction technique [1,2], we derive the exact firing rate equations for a population of identical neurons, and we study numerically and analytically the phase diagrams for both excitatory and inhibitory coupling. For inhibitory networks, we detect a novel region of oscillations, called quasiperiodic partial synchronization [3], and relate it with fast neuronal oscillations.

Network description

Quadratic integrate-and-fire (QIF) neuron model

- $\tau \dot{V}_i = V_i^2 + I_i$

  - τ: Membrane time costant
  - V_i^2: **Membrane Potential**
  - I_i: Input current



- If $V_i \geq V_{threshold}$ then $V_i \leftarrow V_{reset}$

N all-to-all coupled QIF neurons with delay

$$I_i = \eta_i + J s_D$$

- D: Time delay
- η_i: Characterizes neurons intrinsic dynamics
  - $\eta_i < 0$ — Excitable Neurons: Neurons are quiescent, they don't emit spikes spontaneously.
  - $\eta_i > 0$ — Self-sustained oscillators: Neurons fire also without any external input.
- $J s_D$: Recurrent delayed synaptic activation with weight J
  - $J < 0$: **Inhibitory networks**
  - $J > 0$: **Excitatory networks**

$$s_D = \frac{\tau}{N\tau_s}\sum_{j=1}^{N}\sum_k \int_{t-D-\tau_s}^{t-D} \delta(t' - t_j^k)dt'$$

The synaptic activation measures the intensity of the interaction; it counts the number of spikes that are fired at a certain time. We study the case of delayed interaction, where the emitted spikes affect the network after a time delay D

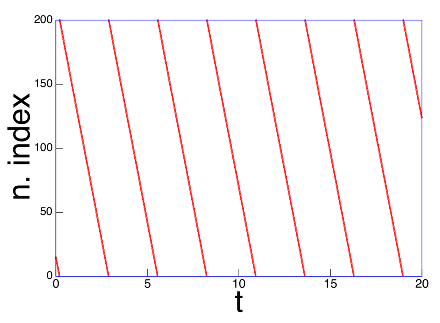
Numerical simulations of the network

Which are the dynamical states we observe in the network of QIF neurons?

- Incoherent/Splay state. Neurons fire asynchronously
- Synchronous state. Neurons fire in phase.
- For Excitable neurons: Global quiescent state. The mean activity of the network is zero.
- For Inhibitory networks (J<0): **Novel partially synchronized state**.

Raster plots

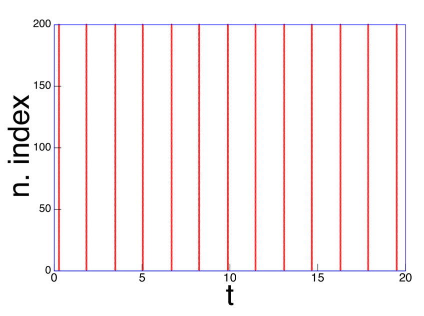
Incoherent state



$J = 1 \quad \sqrt{\bar{\eta}} = 1$

Neurons keep firing at different points.

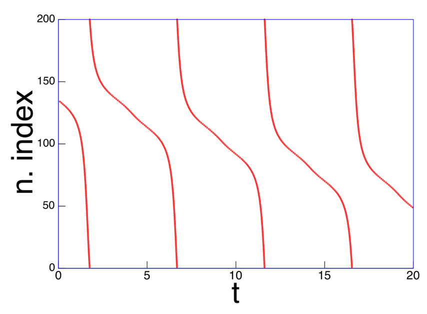
Synchronized state



$J = 5 \quad \sqrt{\bar{\eta}} = 2.5$

Neurons are in phase, they fire always at the same time.

Partially synchronized state

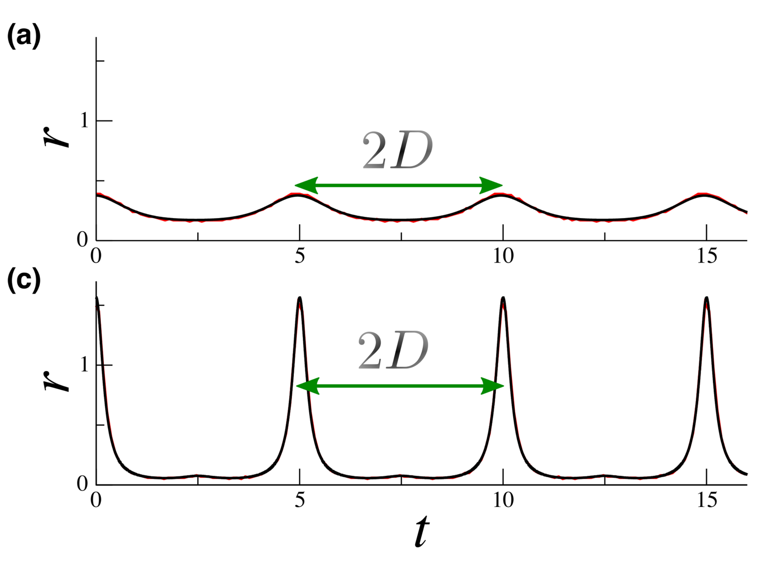


$J = -1.85 \quad \sqrt{\bar{\eta}} = 2.5$

The firing order is preserved, but the dynamics of the single units is not fully periodic.

Partially Synchronized state

We analyze the partially synchronized state in the network of QIF neurons



$J = -1.65 \quad \sqrt{\bar{\eta}} = 2.5$

$J = -1.85 \quad \sqrt{\bar{\eta}} = 2.5$

- Oscillations period: $T = 2D \xrightarrow{D \sim 5 \text{ ms}} f \sim 100 \text{ Hz}$ **Fast Brain Oscillations**

Firing rate equations (FREs)

Derivation of mean field equations

Assumptions

- Thermodynamic limit: $N \to \infty$
- $V_{\text{th}} \to \infty$ and $V_{\text{reset}} \to -\infty$
- Lorentzian distribution of voltages with width Δ and mean $\bar{\eta}$
- Infinitely fast synapses: $\tau_s \to 0$

N QIF equations → **2 mean-field equations**

$$\tau \dot{r} = 2rv + \frac{\Delta}{\pi} \qquad \tau \dot{v} = v^2 + \bar{\eta} + \tau J r_D - \tau^2\pi^2 r^2$$

- r: Firing rate of the network (number of spikes per unit time)
- v: Mean Membrane potential of the network
- $\tau J r_D$: Spikes emitted at time t push the voltage up at time t+D
- $\tau^2\pi^2 r^2$: High firing rate → Low voltages

Adimensionalization

- We can set $D = \tau = 1$ without loss of generality through the following rescaling

$$\tilde{t} = \frac{t}{D} \qquad \tilde{r} = rD \qquad \tilde{v} = \frac{D}{\tau}v \qquad \tilde{J} = \frac{D}{\tau}J \qquad \tilde{\bar{\eta}} = \left(\frac{D}{\tau}\right)^2 \bar{\eta} \qquad \tilde{\Delta} = \left(\frac{D}{\tau}\right)^2 \Delta$$

Phase diagrams

Identical Neurons $\Delta = 0$

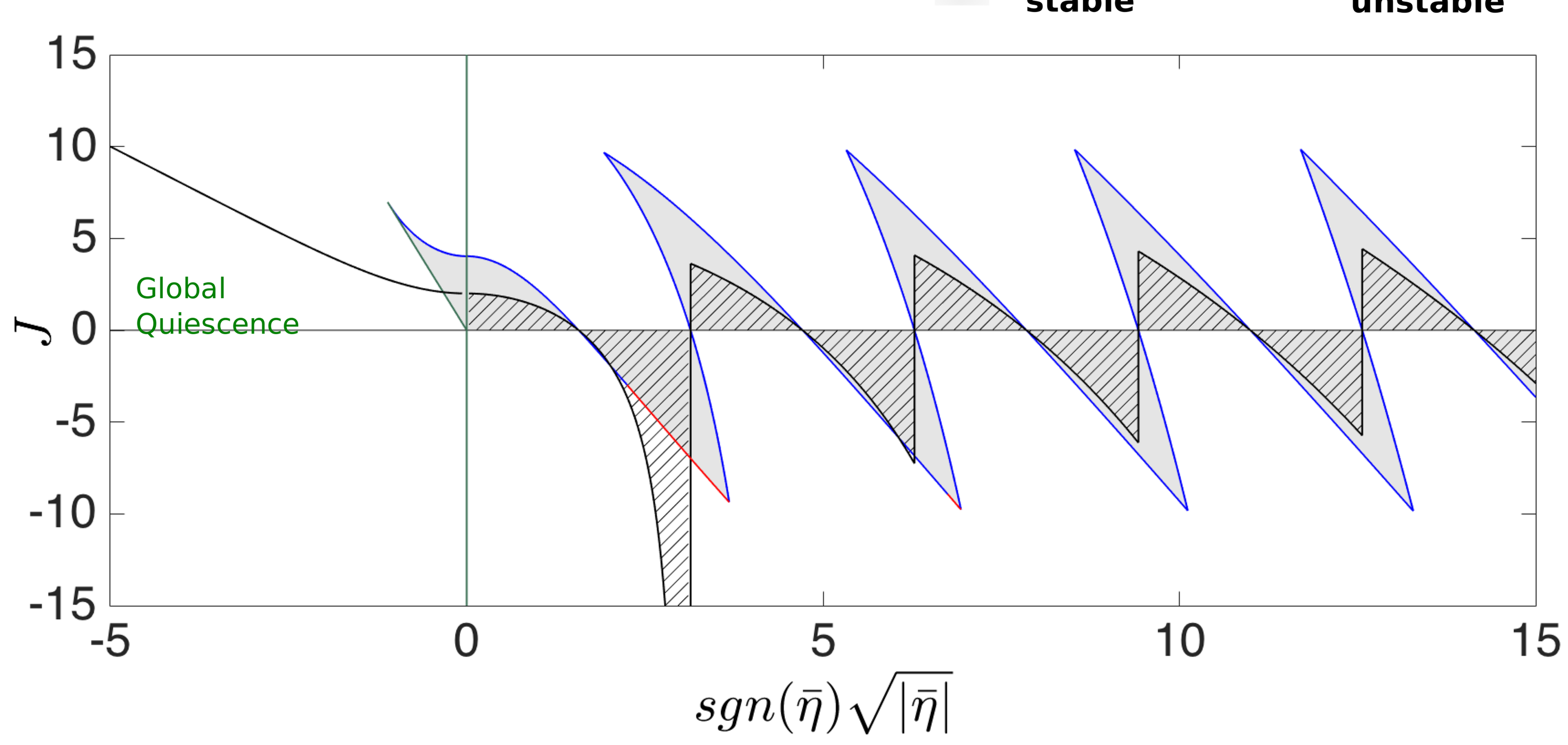
The linear stability analysis of the splay state gives the boundaries:

Incoherence boundaries
$$J_H^{(n)} = \begin{cases} \frac{\pi(\Omega_n^2 - 4\bar{\eta})}{\sqrt{-4\bar{\eta} + 2\Omega_n^2}} & n \text{ even} \\ \frac{\pi(\Omega_n^2 - 4\bar{\eta})}{\sqrt{6\Omega_n^2 + 12\bar{\eta}}} & n \text{ odd} \end{cases} \quad \text{with} \quad \Omega_n = n\pi$$

With an angular transformation of variables, we can derive full synch boundaries

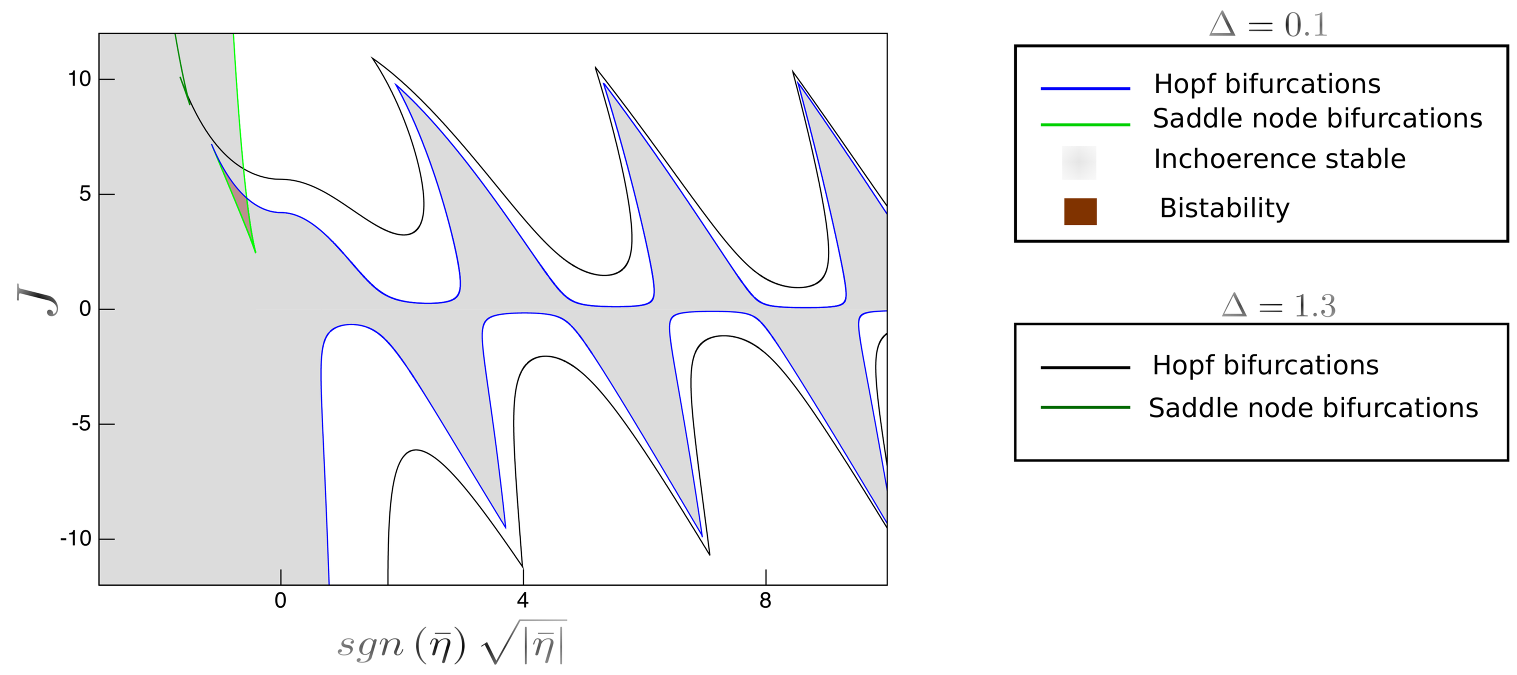
Synchronization boundaries
$$J_c^{(n')} = 2\sqrt{\bar{\eta}}\cot\left(\frac{\sqrt{\bar{\eta}}}{n'}\right) \qquad n' \quad \text{odd}$$

Phase diagram



Green lines: saddle node bifurcations. Blue and red lines: sub-critical and super-critical Hopf bifurcations respectively. When the bifurcation is supercritical, a small amplitude limit cycle is created, the partially synchronized state.

Effect of Heterogeneities ($\Delta \neq 0$)



Conclusions

- We derived the exact firing rate equations for a network of a population of all-to-all coupled QIF neurons with delayed synaptic interactions.
- The interplay between inhibition and synaptic delay is confirmed to be an important mechanism of generation of complex oscillatory pattern.
- We reported the existance of a new oscillatory state for inhibitory coupling, that can be related with fast brain oscillations

Bibliography

[1] E. Montbrió, D. Pazó and A. Roxin. Phys Rev X 2015.
[2] D. Pazó and E. Montbrió, *submitted*.
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